Testing a symbolic program for computer calculating of second order covariant derivatives

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1 Introduction

A symbolic computer program which analytically calculates second and higher order covariant derivatives of arbitrary rank tensors may be tested by applying it to various tensor expressions involving the derivatives where the outcome of the calculation may be relatively easily checked „by hand" or where the outcome is already known. These expressions include differential identities for the Riemann tensor and equations for a Killing vector field.

Notation:
$a, b, c, d, \ldots = 0, 1, 2, 3$ – indices of spacetime coordinates.
$g_{ab}(x^c)$ – spacetime metric of signature $(-, +, +, +)$.
$:=$ – defintion, $\equiv$ – identically equals to,
$\nabla_a T \equiv T_{;a}$ – the covariant derivative with respect to the metric.
The conventions for the curvature tensor are as in Hawking & Ellis' book:

$$R^a{}_{bcd} := \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^m_{db}\Gamma^a_{cm} - \Gamma^m_{cb}\Gamma^a_{dm}, \qquad R_{bd} = R^a{}_{bad}.$$

Terms containing second covariant derivatives:

$$R_{cadb}{}^{;cd} \equiv \nabla^d \nabla^c R_{cadb} \equiv g^{de} g^{cf}\, \nabla_e \nabla_f\, R_{cadb},$$

$$R_{ab;c}{}^{;c} \equiv \nabla^c \nabla_c R_{ab} \equiv g^{cd}\, \nabla_d \nabla_c\, R_{ab},$$

$$R_{;ab} \equiv \nabla_b \nabla_a R := \nabla_b(\partial_a R), \qquad R^c_{a;bc} \equiv \nabla_c \nabla_b R^c_a.$$

2 Identities for the curvature tensor following from the generalized Bianchi identity valid for any dimension ≥ 3

One defines two tensors which vanish identically for the Riemann tensor,

$$A_{ab} := R_{cadb}{}^{;cd} - R_{ab;c}{}^{;c} + R_{ac}R^c_b + R_{cabd}R^{cd} + \frac{1}{2}R_{;ab} \equiv 0, \tag{1}$$

$$C_{ab} := R^c{}_{a;bc} + R^c{}_{b;ac} - 2R_{acdb}R^{cd} - 2R_{ac}R^c_b - R_{;ab} \equiv 0. \tag{2}$$

The program should check that the tensors A_{ab} and C_{ab} vanish identically for some known solutions to Einstein field equations *inside* matter ($R_{ab} \neq 0$):
a) perfect fluid, then $T_{ab} = (\rho + p)u_a u_b + pg_{ab}$ where $u^a u_a = -1$,
b) the electromagnetic field, $T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}$, where $F^{ab}{}_{;b} = 0$,
c) pure radiation field (null field or ultrarelativistic dust), $T_{ab} = \phi^2(x^c)k_a k_b$ with $k^a k_a = 0$.
None of these energy–momentum tensors will explicitly appear in the identities and their role is solely to ensure that Ricci tensor does not vanish and identities (1) and (2) are not exceedingly simplified (if $R_{ab} = 0$ then (1) is reduced to one term and in (2) all the terms vanish separately).
The following spacetime metrics are taken from H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Univ. Press, Cambridge (2003).

1. Homogeneous spacetime generated by a homogeneous nonnull electromagnetic field, this is Bertotti–Robinson spacetime, $x^a = (t, x, \theta, \phi)$,

$$ds^2 = -\sinh^2 x\, dt^2 + dx^2 + d\theta^2 + \sin^2\theta\, d\phi^2. \tag{3}$$

The metric is conformally flat and spherically symmetric, the field F^{ab} is sourceless. Accordingly the spacetime has 4 Kiling vector fields: the timelike $K^a_t = \delta^a_0$ and the 3 rotational Killing vectors of the ordinary sphere S^2, e. g. the generator of rotations about x axis is $K^a_x = (0, 0, -\sin\phi, -\mathrm{ctg}\theta\cos\phi)$.

2. The Gödel spacetime ($x^a = (t, x, y, z)$):

$$ds^2 = -dt^2 - 2e^x\, dt\, dz + dx^2 + dy^2 - \frac{1}{2}e^{2x}\, dz^2. \tag{4}$$

The source is pressureless dust. The metric is symmetric with respect to translations along the axes t, y and z, i. e. $K^a_t = \delta^a_0$, $K^a_y = \delta^a_2$, $K^a_z = \delta^a_3$.

3. Open Robertson–Walker spacetime ($x^a = (t, r, \theta, \phi)$):

$$ds^2 = -dt^2 + a^2(t)[dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta\, d\phi^2)], \tag{5}$$

a perfect fluid is the source. There are 6 Killing vectors including the 3 rotational Killing fields of S^2; here again one can take the K^a_x field.

4. Reissner–Nordström spacetime with the cosmological constant, $x^a = (t, r, \theta, \phi)$,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{6}$$

the matter is the electrostatic field due to the electric charge Q. The metric admits 4 Killing vector fields: the generator of translations in time, $K^a_t = \delta^a_0$ and the 3 generators of rotations of S^2.

5. The Bell–Szekeres solution in coordinates $x^a = (u, v, x, y)$:

$$ds^2 = -2du\, dv + \cos^2(Au - Bv)dx^2 + \cos^2(Au + Bv)dy^2, \tag{7}$$

where A and B are constant. This spacetime represents the collision of two electromagnetic shock waves generating impulsive gravitational waves. The metric is invariant under translations along the x and y axes, $K^a_x = \delta^a_2$ and $K^a_y = \delta^a_3$.

6. The Robinson–Trautman spacetime, $x^a = (u, r, x, y)$,

$$ds^2 = -2H(u, r, x, y)\, du^2 - 2du\, dr + \frac{r^2}{P^2}(dx^2 + dy^2), \tag{8}$$

where $P(u, x, y)$ is independent of r. The functions H and P satisfy some differential equations since the spacetime is generated by a kind of electromagnetic field. In the test one assumes that $H(u, r, x, y)$ and $P(u, x, y)$ are arbitrary. Symmetries appear only for special solutions H and P.

7. The Kundt's class of solutions in $x^a = (u, v, x, y)$ has the metric

$$\begin{aligned} ds^2 &= -2H(u, v, x, y)du^2 - 2du\, dv - 2U(u, v, x, y)du\, dv - 2V(u, v, x, y)du\, dy + \\ &+ \frac{1}{P^2}(dx^2 + dy^2), \end{aligned} \tag{9}$$

where $P = P(u, x, y)$ does not depend on v. Sources are electromagnetic and radiation fields and perfect fluid for $p = -\rho$. In general there are no symmetries, only for special solutions.

8. Van Stockum spacetime generated by a rotating dust, $x^a = (t, r, \phi, z)$,

$$ds^2 = -dt^2 - 2Ar^2\, dt\, d\phi + r^2(1 - A^2r^2)d\phi^2 + e^{-A^2r^2}(dr^2 + dz^2), \tag{10}$$

A =const. Spacetime is stationary and has the cylindrical symmetry. It possesses 3 Killing fields: $K_t^a = \delta_0^a$, $K_\phi^a = \delta_2^a$ and $K_z^a = \delta_3^a$.

9. Kerr–Newman spacetime of a rotating charged black hole, $x^a = (t, r, \theta, \phi)$,

$$\begin{aligned} ds^2 &= \left(-1 + \frac{2Mr - Q^2}{\rho^2}\right) dt^2 - \frac{2A}{\rho^2}(2Mr - Q^2)\sin^2\theta\, dt\, d\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2\, d\theta^2 + \\ &+ \left[r^2 + A^2 + \frac{A^2}{\rho^2}(2Mr - Q^2)\sin^2\theta\right]\sin^2\theta\, d\phi^2, \end{aligned} \tag{11}$$

where

$$\rho^2 = r^2 + A^2\cos^2\theta \qquad \text{and} \qquad \Delta = r^2 - 2Mr + A^2 + Q^2; \tag{12}$$

$A = M/J$ is the angular momentum per unit mass and Q is the electric charge of the black hole. There are 2 independent Killing fields generating the stationarity and cylindrical symmetry, $K_t^a = \delta_0^a$ and $K_\phi^a = \delta_3^a$. For the testing purposes it is convenient to apply their special linear combination representing the null Killing field generating the event horizon,
$K_\Omega^a = \delta_0^a + \Omega\delta_3^a = (1, 0, 0, \Omega)$,
where

$$\Omega = \frac{A}{r_+^2 + A^2} \qquad \text{and} \qquad r_+ = M + \sqrt{M^2 - A^2 - Q^2}.$$

Actually instead of Ω one may take any constant, yet is it convenient to apply the angular velocity of the horizon as the coefficient.

3 Identities for the curvature tensor valid only for dimension $d = 4$

In dimension $d = 4$ one defines the Bach tensor (up to a numerical factor):

$$B_{ab} := R_{;ab} + \frac{1}{2}g_{ab}R_{;c}{}^{;c} - 3R_{ab;c}{}^{;c} + 2RR_{ab} - \frac{1}{2}R^2 g_{ab} + 6R_{cabd}R^{cd} + \frac{3}{2}g_{ab}R^{cd}R_{cd}. \tag{13}$$

I. The Bach tensor is divergence–free:

$$\nabla^b B_{ab} \equiv g^{bc}\nabla_c B_{ab} \equiv 0. \tag{14}$$

The program should check the identity (14) for the set of 9 solutions to Einstein field equations in matter given in section 1.

II. The Bach tensor is conformally invariant in $d = 4$:

$$\text{if} \qquad \bar{g}_{ab} = \Omega^2(x^c) g_{ab}, \quad \text{then} \quad B_{ab}(\bar{g}_{cd}) = \Omega^{-2} B_{ab}(g_{cd}). \tag{15}$$

a) As a special and very simple check the program should prove that for any conformally flat spacetime, $\bar{g}_{ab} = \Omega^2(t, x, y, z)\eta_{ab}$, where $\eta = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric and Ω is arbitrary, there is

$$B_{ab}(\Omega^2 \eta_{cd}) \equiv 0. \tag{16}$$

b) The program should verify the identity (15) for the set of 9 solutions in matter given in section 1.

c) One can also test the program by verifying the id. (15) for a class of known *vacuum* solutions to Einstein field equations. In this case $R_{ab}(g_{cd}) = 0$ and the Bach tensor is reduced to $B_{ab}(g_{cd}) = 0$, while $B_{ab}(\bar{g}_{cd})$ is given by the full expression (13) and satisfies $B_{ab}(\bar{g}_{cd}) = 0$.
We propose the following two vacuum spacetimes to this purpose.

10. Bianchi type I cosmological spacetime, $x^a = (t, x, y, z)$,

$$ds^2 = -dt^2 + t^{2p_1}\, dx^2 + t^{2p_2}\, dy^2 + t^{2p_3}\, dz^2, \tag{17}$$

p_1, p_2, and p_3 are constant and are expressed in terms of an arbitrary parameter $u \geq 1$,

$$p_1 = \frac{-u}{1 + u + u^2}, \qquad p_2 = \frac{1 + u}{1 + u + u^2}, \qquad p_3 = \frac{u(1 + u)}{1 + u + u^2}. \tag{18}$$

The spacetime has 3 Killing fields corresponding to translations along the spatial axes, $K^a_x = \delta^a_1$, $K^a_y = \delta^a_2$ and $K^a_z = \delta^a_3$.

11. Plane–fronted gravitational wave with parallel rays (*pp–wave*), $x^a = (u, v, x, y)$,

$$ds^2 = -2[(x^2 - y^2)\cos(2ku) - 2xy \sin(2ku)]\, du^2 - 2du\, dv + dx^2 + dy^2, \tag{19}$$

k =const. This metric has a 6-dimensional isometry group, here we single out the null Killing vector $K^a_v = \delta^a_1$ and a vector corresponding to this particular form of g_{00}, it is equal to $K^a_p = (1, 0, ky, -kx)$. The length of the latter vector is indefinite – it may be timelike or spacelike.

4 Identities for a Killing vector field

Any Killing field satisfies equations $\nabla_a K_b + \nabla_b K_a = 0$. From these the following identity arises:

$$D_{abc} := K_{a;bc} - R_{abcd}\, K^d \equiv 0. \tag{20}$$

The trace of (20) is $K_{a;c}{}^{;c} + R_{ad}K^d \equiv 0$ and applying the covariant derivative ∇_b to it one gets

$$E_{ab} := g^{cd} K_{a;cdb} + R_{ac;b}\, K^c - R^c_a\, K_{b;c} \equiv 0. \tag{21}$$

Almost all known solutions to Einstein field equations in vacuum or in matter possess some symmetries, that is, they have at least one Killing vector. In fact, among 11 solutions cited above, only Robinson–Trautman and Kundt's class of spacetimes (metrics 6 and 7) have no symmetries (and special solutions in these two classes do have symmetries, but these are so complicated that are impractical for our purpose). Then the computer program may be tested on identities (20) and (21) for the remaining 9 spacetimes.

It should be noticed that for a stationary metric, $(\partial/\partial x^0) g_{ab} = 0$, one has $K^a = \delta^a_0$ and the first term in (20) is

$$K^a{}_{;bc} = \partial_c \Gamma^a_{b0} + \Gamma^a_{cd}\Gamma^d_{b0} - \Gamma^a_{d0}\Gamma^d_{cb}. \tag{22}$$

Comment

All these identities have been suitably chosen to test computations of higher (second and third) order covariant derivatives by a symbolic program. Actually these tests may be equally well applied to *any* program for symbolic tensor calculations.

Acknowledgments

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References

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