Wave Equation: Disk

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The vibrations of a circular drumhead and the behavior of an electromagnetic wave or a quantum-mechanical particle confined to a circular region can be understood by solving the wave equation in polar coordinates on a disk. Let $\Psi(r, \theta, t)$ be the wave function (displacement of the drumhead, electric or magnetic field of the electromagnetic wave, Schrödinger wave function of the particle), which is a function of the polar coordinates r and θ and of the time t. The wave is confined to a disk of radius R and has boundary condition $\Psi(R, \theta, t) = 0$ for all θ and t. Another condition is that Ψ must be regular (not infinite) at the origin; and finally there is a continuity condition that $\Psi(r, \theta, t) = \Psi(r, \theta + 2\pi, t)$ for all r, θ, and t.

Let v be the speed of waves on the disk; then the wave equation is

$$\nabla^2 \Psi(r, \theta, t) = \frac{1}{v^2} \frac{\partial^2 \Psi(r, \theta, t)}{\partial t^2}. \tag{1}$$

The two-dimensional Laplacian in polar coordinates looks like the 3-D Laplacian in cylindrical coordinates without a z term, so the wave equation becomes

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}. \tag{2}$$

We will use the technique of separation of varaiables to look for solutions of the form

$$\Psi(r, \theta, t) = f(r) g(\theta) h(t). \tag{3}$$

Substituting this ansatz into the wave equation, we have

$$g(\theta) h(t) \frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{g(\theta) h(t)}{r} \frac{\mathrm{d}f(r)}{\mathrm{d}r} + \frac{f(r) h(t)}{r^2} \frac{\mathrm{d}^2 g(\theta)}{\mathrm{d}\theta^2} = \frac{f(r) g(\theta)}{v^2} \frac{\mathrm{d}^2 h(t)}{\mathrm{d}t^2} \tag{4}$$

where all derivatives are now total derivatives. Multiplying through by $v^2 / f(r) g(\theta) h(t)$, we have

$$\frac{v^2}{f(r)} \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{v^2}{r f(r)} \frac{\mathrm{d}f}{\mathrm{d}r} + \frac{v^2}{r^2 g(\theta)} \frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} = \frac{1}{h(t)} \frac{\mathrm{d}^2 h}{\mathrm{d}t^2}. \tag{5}$$

Notice that the right-hand side of this equation is a function only of t, and not or r or θ, and that the left-hand side does *not* depend on t at all. The only way for this to happen is if both sides are actually constant. Let us call this unknown constant $-\omega^2$; then

$$\frac{1}{h(t)} \frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = -\omega^2 \tag{6}$$

and, multiplying through by $h(t)$,

$$\frac{1}{v^2}\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = -\omega^2 h(t). \tag{7}$$

This is the differential equation that describes a sinusoid, so that generally

$$h(t) = A\cos(\omega t) + B\sin(\omega t) \tag{8}$$

or, equivalently,

$$h(t) = C\sin(\omega t + \phi) \tag{9}$$

where A and B or C and ϕ are determined by the initial conditions. A common configuration is $A = 0$ and $B = 1$ (or $C = 1$ and $\phi = 0$) so that $\Psi = 0$ at $t = 0$. Another common configuration is $A = 1$ and $B = 0$, which means that the wave function has its maximum magnitude at $t = 0$; this corresponds to starting the clock when a drum is hit.

We how have

$$\frac{v^2}{f(r)}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{v^2}{rf(r)}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{v^2}{r^2 g(\theta)}\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} = -\omega^2. \tag{10}$$

Multiplying through by r^2/v^2, and defining $k = \omega/v$, we have

$$\frac{r^2}{f(r)}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{r}{f(r)}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{1}{g(\theta)}\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} = -k^2 r^2 \tag{11}$$

where k, which has units of 1/distance, is called the wave number. Gathering all the r terms on the left and the θ terms on the right, we have

$$\frac{r^2}{f(r)}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{r}{f(r)}\frac{\mathrm{d}f}{\mathrm{d}r} + k^2 r^2 = -\frac{1}{g(\theta)}\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2}. \tag{12}$$

Again, each side must be equal to a constant, which we will call n^2. Now we have

$$-\frac{1}{g(\theta)}\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} = n^2 \tag{13}$$

or

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} = -n^2 g(\theta) \tag{14}$$

which is, again, the equation for a sinusoid of the form

$$g(\theta) = D\cos(n\theta) + E\sin(n\theta) \tag{15}$$

or

$$g(\theta) = \sin(n\theta + \theta_0) \tag{16}$$

where we are disregarding an overall multiplicative constant. We impose the continuity condition, which is that $g(\theta) = g(\theta + 2\pi)$, and from it deduce that n must be an integer.

Turning to the radial component of the wave function, we now have

$$\frac{r^2}{f(r)}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{r}{f(r)}\frac{\mathrm{d}f}{\mathrm{d}r} + k^2r^2 = n^2 \tag{17}$$

so that

$$r^2\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + r\frac{\mathrm{d}f}{\mathrm{d}r} + k^2r^2 f(r) = n^2 f(r) \tag{18}$$

or

$$r^2\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + r\frac{\mathrm{d}f}{\mathrm{d}r} + (k^2r^2 - n^2) f(r) = 0. \tag{19}$$

If we change to the dimensionless variable $x = kr$, we have

$$x^2\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + x\frac{\mathrm{d}f(x)}{\mathrm{d}x} + (x^2 - n^2) f(x) = 0 \tag{20}$$

which is a Bessel differential equation, whose solutions are the Bessel functions $J_n(x)$ and $Y_n(x)$. The $Y_n(x)$ are not regular at the origin, and must be disregarded here, so we are left with

$$f(x) = J_n(x) \tag{21}$$

or, switching back to using r,

$$f(r) = J_n(kr) \tag{22}$$

where we are again disregarding an overall multiplicative constant.

The boundary condition, $f(R) = 0$, imposes a restriction on k. If we denote by $j_{n,m}$ the mth positive zero of $J_n(x)$, then we must have

$$kR = j_{n,m} \tag{23}$$

for some positive integer m, so that

$$k = \frac{j_{n,m}}{R} \tag{24}$$

and

$$\omega = kv = \frac{v j_{n,m}}{R}. \tag{25}$$

Putting it all together, we have

$$\Psi_{n,m}(r,\theta,t) = C J_n\left(\frac{j_{n,m} r}{R}\right)\sin(n\theta + \theta_0)\sin\left(\frac{v j_{n,m} t}{R} + \phi\right) \tag{26}$$

where C, ϕ, and θ_0 are determined by the initial conditions.

Any linear combination of $\Psi_{n,m}$'s is also a solution to the original wave equation, so we can write a general solution as a Fourier-Bessel series

$$\Psi(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m}\Psi_{n,m}(r,\theta,t) \tag{27}$$

and, because the Bessel functions are orthogonal, we can solve for the coefficients $c_{n,m}$ using Fourier analysis.

Physically, ω is related to the frequency of oscillation f by $\omega = 2\pi f$, so that the frequency of a mode is

$$f_{n,m} = \frac{\omega_{n,m}}{2\pi} = \frac{v j_{n,m}}{2\pi R} \tag{28}$$

and the period T is

$$T_{n,m} = \frac{1}{f_{n,m}} = \frac{2\pi R}{v j_{n,m}}. \tag{29}$$

The frequencies are not integer multiples of the lowest frequency, so drums are not harmonic in the musical sense.