SOME ELEMENTARY THEORY OF CONVEX POLYHEDRA

CHRISTOPHER J. HENRICH

Abstract. An exposition of some basic theorems about convex polyhedra, with some attention to computational issues.

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"Hobbits delighted ... to have books filled with things that they already knew, set out fair and square with no contradictions."
– J. R. R. Tolkien *The Fellowship of the Ring*, Prologue, §1 at end.

1. Introduction

The theory of convex polyhedra in several dimensions is important in a wide variety of contexts. For instance, it underlies the topic of linear programming. The text by Ziegler[3] is modern and inclusive. Among the publicly available collections of software that handle convex polyhedra, the Parma Polyhedra Library (PPL) at http://www.cs.unipr.it/ppl/ is very well designed and documented.

For another project, I wanted to compute some elementary properties of a convex polyhedron. Specifically, I had a convex polytope, defined by equalities and inequalities. I wanted to project it along one or more coordinate axes, and characterize the resulting polytope in a similar way. Ziegler showed me the theory, but omitted some details and relegated others to exercises.[1] The PPL dealt with all the computational issues, but was dauntingly large. I decided to do it myself.

The process of explaining it to myself, well enough that I adequately knew what I was doing, has resulted in these notes. To a large extent they follow Ziegler; the algorithmic aspects are strongly influenced by the PPL, and a crucial proof comes from Komornik[2].

A convex polytope can also be characterized by its vertices. The two characterizations can be used together to good effect. Therefore a considerable part of these notes is devoted to the relationship between these two characterizations.

Convex polytopes are bounded, and thus are a special case of convex polyhedra, which may be unbounded. I found that another special case, that of convex polyhedral cones, has two advantages: its properties are somewhat simpler, and the general case can be reduced to this special case. Therefore, much of these notes is focused on convex polyhedral cones.

1.1. **Notation.** In these notes, $\mathbb{R}^d$ represents the vector space of all d-dimensional column vectors with real components. Vectors will be named in bold-face lower case, e.g. $\mathbf{x}$. When it is necessary to be explicit about the components of a particular vector, I will write $(v_1, v_2, \ldots, v_d)^T$. The standard basis of $\mathbb{R}^d$ is $\mathbf{e}_1, \ldots, \mathbf{e}_d$. The

[1] So, to some extent, these notes record my working of those exercises. I do not feel that I am supplying anyone with a spoiler or cheat-sheet, however, because I had to do the exercises for a different special case than the one that Ziegler prefers.

usual norm and inner product will be denoted $\| \dots \|$ and $(\dots, \dots)$; thus, $(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \mathbf{w}$.

If $\mathbf{x} \in \mathbb{R}^d$, then the **ray generated by x** is

$$R_{\mathbf{x}} = \{t\mathbf{x} | t \geq 0\}$$

and the **line generated by x** is

$$L_{\mathbf{x}} = \{t\mathbf{x} | t \in \mathbb{R}\} .$$

We shall use sans-serif capitals to name lists of vectors, for instance: $\mathsf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_m)$. If $\mathbf{a}_\iota \in \mathbb{R}^d, \iota = 1, \dots, m$, then $\mathsf{A} \in \mathbb{R}^{d \times m}$. A list of m real numbers is a row vector, or a member of $\mathbb{R}^{1 \times m}$, denoted by a small sans-serif letter. The empty list, or list of length 0, will be denoted O.

1.2. **Two definitions of "convex polyhedron".** In general, a set $S \subseteq \mathbb{R}^d$ is **convex** if, whenever two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to S, so do the points of the segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$. Algebraically, these are the vectors $t_1\mathbf{x}_1 + t_2\mathbf{x}_2$, $t_1 \geq 0$, $t_2 \geq 0$, $t_1 + t_2 = 1$. A set S is a **convex cone** if it is convex and, for every $\mathbf{x} \in S$ and $t \geq 0$, $t\mathbf{x} \in S$. Algebraically, S is a convex cone if, given $\mathbf{x}_1 \in S$ and $\mathbf{x}_2 \in S$, and $t_1 \geq 0$, $t_2 \geq 0$,

$$t_1\mathbf{x}_1 + t_2\mathbf{x}_2 \in S .$$

These notes are about convex sets that are "polyhedral". Part of our task is to define and describe polyhedral convex sets. We shall give two descriptions, which we must show are equivalent.

A closed convex polyhedron may be defined as the intersection of a finite set of closed half-spaces in $\mathbb{R}^d$. This may or may not be bounded, and it may or may not be contained in an affine subspace such as a hyperplane. The definition includes as extreme cases the empty set and the entirety of $\mathbb{R}^d$. This definition can be made more formal:

Definition 1.1. Let $\mathsf{A} \in \mathbb{R}^{d \times m}$, $\mathsf{B} \in \mathbb{R}^{d \times n}$, $\mathsf{v} \in \mathbb{R}^{1 \times m}$, and $\mathsf{w} \in \mathbb{R}^{1 \times n}$. The ***H*-polyhedron presented by** A, B, v, and w is the set

$$\mathrm{Poly}\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{v} & \mathsf{w} \end{pmatrix} = \\ \{\mathbf{x} \in \mathbb{R}^d | (\mathbf{a}_\alpha, \mathbf{x}) = v_\alpha, \alpha = 1, \dots, m; (\mathbf{b}_\beta, \mathbf{x}) \leq w_\beta, \beta = 1, \dots, n\}$$

where

$$\begin{aligned} \mathsf{A} &= (\mathbf{a}_1, \dots, \mathbf{a}_m), \\ \mathsf{B} &= (\mathbf{b}_1, \dots, \mathbf{b}_n), \\ \mathsf{v} &= (v_1, \dots, v_m), \\ \mathsf{w} &= (w_1, \dots, w_n). \end{aligned}$$

If v and w are both 0, we shall write $\mathrm{Poly}(\mathsf{A}, \mathsf{B})$; this is the ***H*-cone presented by** A and B.

Definition 1.2. A **closed convex polyhedron** is any set P which can be presented as an H-polyhedron. The **dimension** of P, denoted $\dim(P)$, is the dimension of the smallest affine subspace containing P.

Another way to describe a convex polyhedron is to construct it from generating elements. These elements may be lines, rays, or points.

Definition 1.3. Let $\mathsf{A} \in \mathbb{R}^{d\times m}$, $\mathsf{B} \in \mathbb{R}^{d\times n}$, and $\mathsf{C} \in \mathbb{R}^{d\times p}$; let

$$\begin{aligned}(1.1)\qquad &\mathrm{Hull}(\mathsf{A},\mathsf{B},\mathsf{C}) = \\ &\quad\{r_1\mathbf{a}_1 + \cdots + r_m\mathbf{a}_m + s_1\mathbf{b}_1 + \cdots + s_n\mathbf{b}_n + t_1\mathbf{c}_1 + \cdots + t_p\mathbf{a}_p| \\ &\quad r_\alpha \in \mathbb{R}, \alpha = 1,\ldots,m;\ s_\beta \geq 0, \beta = 1,\ldots,n; \\ &\quad t_\gamma \geq 0, \gamma = 1,\ldots,p;\ t_1 + \cdots t_p = 1\}.\end{aligned}$$

Then $\mathrm{Hull}(\mathsf{A},\mathsf{B},\mathsf{C})$ is the **convex-conical hull generated by** the lines $L_{\mathbf{a}_1}$, …, $L_{\mathbf{a}_m}$, the rays $R_{\mathbf{b}_1}$, …, $R_{\mathbf{b}_n}$, and the points $\mathbf{c}_1$,…, $\mathbf{c}_p$, where

$$\begin{aligned}\mathsf{A} &= (\mathbf{a}_1,\ldots,\mathbf{a}_m),\\ \mathsf{B} &= (\mathbf{b}_1,\ldots,\mathbf{b}_n),\\ \mathsf{C} &= (\mathbf{c}_1,\ldots,\mathbf{c}_p).\end{aligned}$$

If the list $\mathsf{C} = \mathsf{O}$, we shall write $\mathrm{Hull}(\mathsf{A},\mathsf{B})$; this is the **cone generated by** the lines $L_{\mathbf{a}_1}$, …, $L_{\mathbf{a}_m}$ and the rays $R_{\mathbf{b}_1}$, …, $R_{\mathbf{b}_n}$.

1.3. **Summary of further sections.** In §2, I define convex polyhedral cones. I show how questions about the more general class of convex polyhedra can be reduced to this subclass.

§3 introduces polarity for closed convex polyhedral cones. §4 introduces the operations of intersection and projection for closed convex polyhedral cones. These can be implemented by an important algorithm called **Fourier-Motzkin elimination**. They are then used in §5, to prove that H-polyhedra and convex-conical hulls are two definitions of the same class of sets. This will, in fact, be the "main theorem" of these notes.

The next part of these notes is motivated by the problem of computing descriptions of closed convex polyhedra. §6 introduces double descriptions of these polyhedra, and describes some of the ways in which these descriptions can be otimized. §7 is concerned with the last stage of optimization, which is to eliminate redundancy from the descriptions. We have to characterize "redundancy," both geometrically and algebraically. In §8 we show how redundant parts of a description can be detected economically.

The final part of these notes is concerned with the problem of computing the double descriptors of convex polyhedral cones. In §9, we consider the effects of adding a single descriptor to part of the double description. In the course of this, we revisit the Fourier-Motzkin algorithm, in which we can not only detect redundant descriptors but prevent their formation. Finally, in §10, we discuss how the contents of these notes are used in some Mathematica(TM) code which these notes accompany.

2. Reduction to the subproblem of convex polyhedral cones

It is useful to define two embeddings of $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$:

$$\iota : (x_1,\ldots,x_d)^T \mapsto (x_1,\ldots,x_d,1)^T$$

and

$$\iota_0 : (x_1,\ldots,x_d)^T \mapsto (x_1,\ldots,x_d,0)^T\ .$$

Definition 2.1. Let S be a closed set in $\mathbb{R}^d$. Then $\mathcal{C}(S)$, the **closed cone** over S, is the closure of the union of the rays $R_{\iota\mathbf{x}}$ for $\mathbf{x} \in S$.

We will show that an H-polyhedron or a convex-conical hull is determined by the closed cone over it.

2.1. Closed cones over H-polyhedra.

Proposition 2.2. *Let P be a non-empty H-polyhedron in $\mathbb{R}^d$:*

$$P = \mathrm{Poly}\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{v} & \mathsf{w} \end{pmatrix} .$$

where

$$\begin{aligned} \mathsf{A} &= (\mathbf{a}_1, \ldots, \mathbf{a}_m) \in \mathbb{R}^{d\times m}, \\ \mathsf{B} &= (\mathbf{b}_1, \ldots, \mathbf{b}_n) \in \mathbb{R}^{d\times n}, \\ \mathsf{v} &= (v_1, \ldots, v_m) \in \mathbb{R}^{1\times m}, \\ \mathsf{w} &= (w_1, \ldots, w_n) \in \mathbb{R}^{1\times n}. \end{aligned}$$

Let

$$\begin{aligned} \tilde{\mathbf{a}}_\alpha &= \iota_0 \mathbf{a}_\alpha - v_\alpha \mathbf{e}_{d+1},\ \alpha = 1, \ldots, m, \\ \tilde{\mathbf{b}}_\beta &= \iota_0 \mathbf{b}_\beta - w_\beta \mathbf{e}_{d+1},\ \beta = 1, \ldots, n, \\ \tilde{\mathbf{b}}_{n+1} &= -\mathbf{e}_{d+1}. \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathsf{A}} &= (\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_m) \in \mathbb{R}^{(d+1)\times m}, \\ \tilde{\mathsf{B}} &= (\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_{n+1}) \in \mathbb{R}^{(d+1)\times(m+1)}. \end{aligned}$$

Then

$$\iota P = \mathrm{Poly}(\tilde{\mathsf{A}}, \tilde{\mathsf{B}}) \cap \iota\mathbb{R}^d. \tag{2.1}$$

Also, the closed cone over P is

$$\mathcal{C}(P) = \mathrm{Poly}(\tilde{\mathsf{A}}, \tilde{\mathsf{B}}).$$

Proof. Equation (2.1) is an immediate consequence of the definitions.

Let us show that $\mathcal{C}(P) \subseteq \mathrm{Poly}(\tilde{\mathsf{A}}, \tilde{\mathsf{B}})$. Indeed, if $\mathbf{x} \in P$ then it is immediate that $\iota\mathbf{x} \in \mathrm{Poly}(\tilde{\mathsf{A}}, \tilde{\mathsf{B}})$. Therefore the ray through $\iota\mathbf{x}$ is contained in $\mathrm{Poly}(\tilde{\mathsf{A}}, \tilde{\mathsf{B}})$. But $\mathrm{Poly}(\tilde{\mathsf{A}}, \tilde{\mathsf{B}})$ is a closed set, so it contains the closure of the union of those rays, which is just $\mathcal{C}(P)$.

Next we show that $\mathrm{Poly}(\tilde{\mathsf{A}}, \tilde{\mathsf{B}}) \subseteq \mathcal{C}(P)$, let $\mathbf{y} \in \mathrm{Poly}(\tilde{\mathsf{A}}, \tilde{\mathsf{B}})$. Then $y_{d+1} \geq 0$ because of the last member of $\tilde{\mathsf{B}}$. There are two cases to consider.

(i) If $y_{d+1} > 0$, let $\mathbf{z} = (y_{d+1})^{-1}\mathbf{y}$. Then $\mathbf{z} = \iota\mathbf{x}$ for some $\mathbf{x} \in P$.

(ii) If $y_{d+1} = 0$, then $\mathbf{y} = \iota_0\mathbf{u}$ where $\mathbf{u} \in \mathbb{R}^d$, and

$$\begin{aligned} (\mathbf{a}_\alpha, \mathbf{u}) &= 0,\ \alpha = 1, \ldots, m, \\ (\mathbf{b}_\beta, \mathbf{u}) &\leq 0,\ \beta = 1, \ldots, n. \end{aligned}$$

Let $\mathbf{x}$ be any point in P. We check that $\mathbf{x} + \lambda\mathbf{u} \in P$, so that $\iota\mathbf{x} + \lambda\mathbf{v} \in \mathcal{C}(P)$. Divide by λ and let $\lambda \to \infty$; we have that $\lambda^{-1}\iota\mathbf{x} + \mathbf{v}$ belongs to $\mathcal{C}(P)$ and tends to $\mathbf{v}$. $\square$

The moral of this tale is that the theory of H-polyhedra can be reduced to that of H-cones.

2.2. **Closed cones over convex-conical hulls.** The following proposition, taken from [2], is crucial to Theorem 3.3, known as "Farkas's lemma."

Proposition 2.3. *Let* $\mathsf{Y} = \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\} \in \mathbb{R}^{d\times k}$*, and let* $K = \mathrm{Hull}(\mathsf{O}, \mathsf{Y})$*. Then for every* $\mathbf{x} \in \mathbb{R}^d$ *there is a closest point* $\mathbf{x}'$ *in* K*.*

Proof. The proof is by induction on k. The statement is obvious if $k = 1$. Let $k \geq 2$. There are three cases to consider.

(1) If $\mathbf{x} \in K$ then we may choose $\mathbf{x}' = \mathbf{x}$.

(2) If $\mathbf{x} \notin K$ but $\mathbf{x}$ belongs to the linear hull L of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}$, then by the induction hypothesis we may assume that, for each j, the cone $K_j = \mathrm{Hull}(\mathsf{O}, (\mathbf{y}_1, \dots, \mathbf{y}_{j-1}, \mathbf{y}_{j+1}, \dots, \mathbf{y}_k))$ has a closest point $\mathbf{x}'_j$ to $\mathbf{x}$. Let $\mathbf{x}'$ be the closest point to $\mathbf{x}$ among $\{\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_k\}$. We show that $\|\mathbf{x} - \mathbf{x}'\| \leq \|\mathbf{x} - \mathbf{z}\|$ for every $\mathbf{z} \in K$.

We have

$$\mathbf{x} = \alpha_1 \mathbf{y}_1 + \cdots + \alpha_k \mathbf{y}_k \,,$$

in which $\alpha_j < 0$ for some j because $\mathbf{x} \notin K$. We also have

$$\mathbf{z} = \gamma_1 \mathbf{y}_1 + \cdots + \gamma_k \mathbf{y}_k \,,$$

in which $\gamma_1, \dots, \gamma_k \geq 0$. Let

$$t := \min\{\gamma_j / (\gamma_j - \alpha_j) \mid 1 \leq j \leq k,\ \alpha_j < 0\}\,.$$

Then $0 \leq t < 1$ and the minimum is attained at some index i. Therefore

$$t\alpha_j + (1-t)\gamma_j \geq 0 \text{ for all } j$$

and

$$t\alpha_i + (1-t)\gamma_i = 0\,.$$

The geometrical meaning of these inequalities is that the segment from $\mathbf{x}$ to $\mathbf{z}$ meets the side K_i of K at $t\mathbf{x} + (1-t)\mathbf{z}$. Therefore

$$\|\mathbf{x} - \mathbf{x}'\| \leq \|\mathbf{x} - \mathbf{x}'_i\| \leq \|\mathbf{x} - (t\mathbf{x} + (1-t)\mathbf{z})\| = (1-t)\|\mathbf{x} - \mathbf{z}\|\,;$$

therefore

$$\|\mathbf{x} - \mathbf{x}'\| \leq \|\mathbf{x} - \mathbf{z}\|\,.$$

(3) If $\mathbf{x} \notin L$, then choose an orthonormal basis $\mathbf{f}_1, \dots, \mathbf{f}_l$ of L and set $\tilde{\mathbf{x}} = (\mathbf{x}, \mathbf{f}_1)\mathbf{f}_1 + \dots (\mathbf{x}, \mathbf{f}_l)\mathbf{f}_l$. From steps (1) and (2), there is in K a closest point $\mathbf{x}'$ to $\tilde{x}$. Now $\mathbf{x} - \tilde{\mathbf{x}}$ is orthogonal to L. So, if $\mathbf{z} \in K$, we have

$$\|\mathbf{x} - \mathbf{x}'\|^2 = \|\mathbf{x} - \tilde{\mathbf{x}}\|^2 + \|\tilde{\mathbf{x}} - \mathbf{x}'\|^2 \leq \|\mathbf{x} - \tilde{\mathbf{x}}\|^2 + \|\tilde{\mathbf{x}} - \mathbf{z}\|^2 = \|\mathbf{x} - \mathbf{z}\|^2\,;$$

thus, $\mathbf{x}'$ is also a closest point in K to $\mathbf{x}$. □

Corollary 2.4. *Let* K *be as in Prop. 2.3; then* K *is closed.*

These results, as stated, apply to conical hulls determined by ray generators. Line generators, while very convenient, are not indispensable, as the following proposition shows.

Proposition 2.5. *Let*

$$\mathsf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_m) \in \mathbb{R}^{d\times m}\,,$$
$$\mathsf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n) \in \mathbb{R}^{d\times n}\,,$$

and let

$$\mathsf{C} = (\mathbf{a}_1, \ldots, \mathbf{a}_m, -\mathbf{a}_1, \ldots, -\mathbf{a}_m, \\ \mathbf{b}_1, \ldots, \mathbf{b}_n) \in \mathbb{R}^{d\times(2m+n)} .$$

Then

$$\mathrm{Hull}(\mathsf{A}, \mathsf{B}) = \mathrm{Hull}(\mathsf{O}, \mathsf{C}) .$$

Proof. Each of these hulls contains the generators of the other hull. □

Corollary 2.6. *Let*

$$\mathsf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_m) \in \mathbb{R}^{d\times m} ,$$
$$\mathsf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n) \in \mathbb{R}^{d\times n} ,$$

and let

$$K = \mathrm{Hull}(\mathsf{A}, \mathsf{B}) .$$

Then for every $\mathbf{x} \in \mathbb{R}^d$ there is a closest point $\mathbf{x}'$ in K. Also, K is closed.

Proof. Apply Proposition 2.5 to K; then apply Proposition 2.3 and Corollary 2.4. □

Similarly, equality constraints are not indispensable; one could replace each of them by a pair of inequalities. However, both line generators and equality constraints make the descriptions more concise and more informative about the sets being described.

Proposition 2.7. *Let*

$$\mathsf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_m) \in \mathbb{R}^{d\times m} ,$$
$$\mathsf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n) \in \mathbb{R}^{d\times n} ,$$
$$\mathsf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_p) \in \mathbb{R}^{d\times p} ,$$

and let

$$P = \mathrm{Hull}(\mathsf{A}, \mathsf{B}, \mathsf{C}) .$$

Let

$$\hat{\mathsf{A}} = (\iota_0\mathbf{a}_1, \ldots, \iota_0\mathbf{a}_m) ,$$
$$\hat{\mathsf{B}} = (\iota_0\mathbf{b}_1, \ldots, \iota_0\mathbf{b}_n, \iota\mathbf{c}_1, \ldots, \iota\mathbf{c}_p) .$$

Then

$$\mathcal{C}(P) = \mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}}) \tag{2.2}$$

and

$$\iota P = \mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}}) \cap \iota\mathbb{R}^d . \tag{2.3}$$

Proof. P consists of points of the form given in (1.1). If $\mathbf{x}$ is any point of that form, then $\iota\mathbf{x} \in \mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}})$; that is to say, $\iota P \subseteq \mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}}) \cap \iota\mathbb{R}^d$. Conversely, if $\mathbf{w} \in \mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}})$, then

$$\mathbf{w} = r_1\iota_0\mathbf{a}_1 + \cdots + r_m\iota_0\mathbf{a}_m + s_1\iota_0\mathbf{b}_1 + \cdots + s_n\iota_0\mathbf{b}_n + t_1\iota\mathbf{c}_1 + \cdots + t_p\iota\mathbf{c}_p \tag{2.4}$$

where $r_\alpha \in \mathbb{R}, \alpha = 1,\dots,m,\ s_\beta \geq 0, s = 1,\dots,n,$ and $t_\gamma \geq 0, \gamma = 1,\dots,p$; if moreover $\mathbf{w} \in \iota\mathbb{R}^d$, then $t_1 + \cdots + t_p = 1$. It follows that $\mathbf{w} = \iota\mathbf{x}$ for some $\mathbf{x}$ of the form given in in (1.1). Therefore, $\mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}}) \cap \iota\mathbb{R}^d \subseteq \iota P$, and (2.3) is proved.

To prove that $\mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}}) \subseteq \mathcal{C}(P)$, consider a vector $\mathbf{w} \in \mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}})$, given by (2.4), and let $t = t_1 + \cdots + t_p$. There are two cases to consider:

(i) If $t > 0$, then $t^{-1}\mathbf{w} \in \iota\mathbb{R}^d$, so $t^{-1}\mathbf{w} \in \iota P$ by (2.3); therefore $\mathbf{w} \in \mathcal{C}(P)$.

(ii) If $t = 0$, then $\mathbf{w} = \iota_0\mathbf{u}$, where $\mathbf{u} \in \mathrm{Hull}(\mathsf{A}, \mathsf{B})$. Let $\mathbf{x} \in P$. If $\lambda > 0$, then $\mathbf{x} + \lambda\mathbf{u} \in P$; therefore $\iota\mathbf{x} + \lambda\mathbf{w} \in \mathcal{C}(P)$. Divide by λ and let $\lambda \to \infty$; we have that $\lambda^{-1}\iota\mathbf{x} + \mathbf{w}$ belongs to $\mathcal{C}(P)$ and tends to $\mathbf{w}$.

To prove that $\mathcal{C}(P) \subseteq \mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}})$, we simply observe that $\mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}})$ is closed, by Corollary 2.6, and contains every ray through a point of ιP. □

This proposition shows that the general case of convex-conical hulls can be reduced to that of conical hulls.

3. Polarity

This is a relation between convex polyhedral cones that naturally relates H-cones to conical hulls.

Definition 3.1. Let K be a convex cone in $\mathbb{R}^d$. The **polar** of K is the set K° defined by

$$K^\circ = \{\mathbf{x} \in \mathbb{R}^d \mid (\mathbf{x}, \mathbf{y}) \leq 0 \text{ for all } \mathbf{y} \in K\}.$$

It is easily seen that K° is also a convex cone.

Proposition 3.2. *Let $\mathsf{A} \in \mathbb{R}^{d\times m}$ and $\mathsf{B} \in \mathbb{R}^{d\times n}$; let*

$$K = \mathrm{Hull}(\mathsf{A}, \mathsf{B}).$$

Then

$$K^\circ = \mathrm{Poly}(\mathsf{A}, \mathsf{B}).$$

Proof. Let $\mathsf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_m)$ and $\mathsf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$. The statement that $\mathbf{x} \in \mathrm{Poly}(\mathsf{A}, \mathsf{B})$ is equivalent to

$$(\mathbf{a}_\alpha, \mathbf{x}) = 0, \alpha = 1, \dots, m,$$
$$(\mathbf{b}_\beta, \mathbf{x}) \leq 0, \beta = 1, \dots, n.$$

But these relations are clearly equivalent to $\mathbf{x} \in K^\circ$. □

This easy proposition says that the polar of a conical hull is an H-cone. What about the polar of an H-cone : is it a conical hull? The affirmative answer to this question is a fundamental result, first proved in [1]. Ziegler[3] gives many equivalent formulations, with references to its history. The proof given here, dependent on Proposition 2.3, is due to Komornik[2].

Theorem 3.3. ***(Farkas's Lemma)*** *Let $\mathsf{A} \in \mathbb{R}^{d\times m}$ and $\mathsf{B} \in \mathbb{R}^{d\times n}$, and let*

$$K = \mathrm{Poly}(\mathsf{A}, \mathsf{B}).$$

Then

$$K^\circ = \mathrm{Hull}(\mathsf{A}, \mathsf{B}).$$

Proof. Let

$$\mathsf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_m)\,,$$
$$\mathsf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)\,.$$

Then the statement that $\mathbf{x} \in K$ is equivalent to the inequalities

$$(\mathbf{a}_\alpha, \mathbf{x}) = 0,\ \alpha = 1, \ldots, m\,, (\mathbf{b}_\beta, \mathbf{x}) \le 0,\ \beta = 1, \ldots, n\,. \tag{3.1}$$

Thus it is immediate that

$$\mathrm{Hull}(\mathsf{A}, \mathsf{B}) \subseteq K^\circ\,.$$

Let us prove that, conversely,

$$K^\circ \subseteq \mathrm{Hull}(\mathsf{A}, \mathsf{B})\,.$$

Let $\mathbf{y} \in K^\circ$; that is, $(\mathbf{y}, \mathbf{x}) \le 0$ for all $\mathbf{x} \in \mathbb{R}^d$ satisfying (3.1).

By virtue of Prop. 2.6, we may let $\mathbf{z}$ be a point in $\mathrm{Hull}(\mathsf{A}, \mathsf{B})$ closest to $\mathbf{y}$. We will prove that

$$(\mathbf{a}_\alpha, \mathbf{y} - \mathbf{z}) = 0, \quad \alpha = 1, \ldots, m, \tag{3.2}$$

$$(\mathbf{b}_\beta, \mathbf{y} - \mathbf{z}) \le 0, \quad \beta = 1, \ldots, n \tag{3.3}$$

and

$$(-\mathbf{z}, \mathbf{y} - \mathbf{z}) \le 0. \tag{3.4}$$

Indeed, if one of the equalities (3.2) were not valid, then for sufficiently small $t \in \mathbb{R}$ such that $t(\mathbf{a}_\alpha, \mathbf{y} - \mathbf{z}) > 0$ we would have

$$\|\mathbf{y} - (\mathbf{z} + t\mathbf{a}_\alpha)\|^2 = \|(\mathbf{y} - \mathbf{z}) - t\mathbf{a}_\alpha\|^2 = \|\mathbf{y} - \mathbf{z}\|^2 - 2t(\mathbf{a}_\alpha, \mathbf{y} - \mathbf{z}) + t^2\|\mathbf{b}_\beta\|^2 < \|\mathbf{y} - \mathbf{z}\|^2\,,$$

whereas if one of the inequalities (3.3) or (3.4) were not valid, then for a sufficiently small $t \in (0, 1)$ we would have

$$\|\mathbf{y} - (\mathbf{z} + t\mathbf{b}_\beta)\|^2 = \|(\mathbf{y} - \mathbf{z}) - t\mathbf{b}_\beta\|^2 = \|\mathbf{y} - \mathbf{z}\|^2 - 2t(\mathbf{b}_\beta, \mathbf{y} - \mathbf{z}) + t^2\|\mathbf{b}_\beta\|^2 < \|\mathbf{y} - \mathbf{z}\|^2\,,$$

that is,

$$\|\mathbf{y} - (\mathbf{z} + t\mathbf{b}_\beta)\|^2 < \|\mathbf{y} - \mathbf{z}\|^2\,, \tag{a}$$

or

$$\|\mathbf{y} - (\mathbf{z} - t\mathbf{z})\|^2 = \|(\mathbf{y} - \mathbf{z}) + t\mathbf{z}\|^2 = \|\mathbf{y} - \mathbf{z}\|^2 - 2t(-\mathbf{z}, \mathbf{y} - \mathbf{z}) + t^2\|\mathbf{y}_i\|^2 < \|\mathbf{y} - \mathbf{z}\|^2\,,$$

that is,

$$\|\mathbf{y} - (\mathbf{z} - t\mathbf{z})\|^2 < \|\mathbf{y} - \mathbf{z}\|^2\,; \tag{b}$$

but $\mathbf{z} + t\mathbf{a}_\alpha$, $\mathbf{z} + t\mathbf{b}_\beta$ and $\mathbf{z} - t\mathbf{z} = (1 - t)\mathbf{z}$ belong to $\mathrm{Hull}(\mathsf{A}, \mathsf{B})$, so inequalities (a) and (b) are contrary to the choice of $\mathbf{z}$.

Now by hypothesis the inequalities (3.2) and (3.3) imply that $(\mathbf{y}, \mathbf{y} - \mathbf{z}) \le 0$. Together with (3.4) this implies $(\mathbf{y} - \mathbf{z}, \mathbf{y} - \mathbf{z}) \le 0$. Thus $\mathbf{y} = \mathbf{z}$ and therefore $\mathbf{y} \in \mathrm{Hull}(\mathsf{A}, \mathsf{B})$. $\square$

Proposition 3.2 and Theorem 3.3 show that polarity gives a very exact correspondence between convex cones presented as H-cones and convex cones described as conical hulls. It is worth noting that for cones of either description, $K = K^{\circ\circ}$.

4. Some operations on cones

The intersection of a convex cone with a linear subspace is a convex cone in the subspace. If the original cone is polyhedral, what about the intersection? It is very easy to see that the intersection of an H-cone with a subspace is also an H-cone. For conical hulls, one feels that the intersection ought to be polyhedral, but proving it is not trivial.

If the enclosing vector space is projected onto a vector space of lower dimension, then a convex cone is projected onto a convex cone. Again one may ask, if the original cone is polyhedral, is the image polyhedral? If the original is a conical hull, then it is very easy to see that the image is also a conical hull. For H-cones, the affirmative answer to the question is, again, very plausible but not trivial to prove.

The non-trivial part of these problems is solved by using a process called **Fourier-Motzkin elimination**. We will define it in a way that is slightly more general than what we need here, but will be useful later.

Definition 4.1. Let $\mathsf{B} \in \mathbb{R}^{d\times n}$; let $\mathbf{v} \in \mathbb{R}^d$. Then $FM(\mathsf{B}, \mathbf{v})$ is a list of the following vectors in $\mathbb{R}^d$:

(1) any element $\mathbf{d}$ of D such that $(\mathbf{v}, \mathbf{b}) = 0$;
(2) for any $\mathbf{b}'$ and $\mathbf{b}''$ in D such that $(\mathbf{v}, \mathbf{b}') > 0$ and $(\mathbf{v}, \mathbf{b}'') < 0$, the vector

$$(\mathbf{v}, \mathbf{b}')\mathbf{b}'' + (-(\mathbf{v}, \mathbf{b}''))\mathbf{b}' \; .$$

Fourier-Motzkin elimination is the process of generating $FM(\mathsf{B}, \mathbf{v})$.

Proposition 4.2. *Let* $\mathsf{A} \in \mathbb{R}^{d\times m}$ *and* $\mathsf{B} \in \mathbb{R}^{d\times n}$. *Let* $\mathbf{v} \in \mathbb{R}^d$ *be orthogonal to each element of* A. *Then*

$$\mathrm{Hull}(\mathsf{A}, \mathsf{B}) \cap \mathbf{v}^{\perp} = \mathrm{Hull}(\mathsf{A}, FM(\mathsf{B}, \mathbf{v})) \; .$$

Proof. To check that the right-hand side of this equation is included in the left, it is sufficient to observe that each generator on the right is contained in the Hull on the left, and is orthogonal to $\mathbf{v}$.

Next we must show inclusion in the other direction. Let $\mathbf{x}$ be contained in the set specified on the left-hand side. Then

$$\mathbf{x} \in \mathrm{Hull}(\mathsf{A}, \mathsf{B}) \; ,$$

and also $(\mathbf{v}, \mathbf{x}) = 0$. Therefore

$$\mathbf{x} = \sum_i s_i \mathbf{a}_i + \sum_i t_i \mathbf{b}_i \; , \quad t_i \geq 0 \, ,$$

and also

$$\sum_i t_i(\mathbf{v}, \mathbf{b}_i) = 0 \; .$$

It may be the case that all the products $t_i(\mathbf{v}, \mathbf{b}_i)$ are 0. If so, then

$$\mathbf{x} \in \mathrm{Hull}(\mathsf{A}, \mathsf{B}^0)$$

where B^0 is a list of those elements $\mathbf{b}$ of B such that $(\mathbf{v}, \mathbf{b}) = 0$. But by Definition 4.1, $\mathsf{B}^0 \subseteq FM(\mathsf{B}, \mathbf{v})$.

On the other hand, consider the case that some of the products $t_i(\mathbf{v},\mathbf{b}_i)$ are not zero. Remembering that n is the number of elements in B, let us define

$$\begin{aligned} P &= \{i \,|\, 1 \le i \le n,\, (\mathbf{v},\mathbf{b}_i) > 0\}\,, \\ M &= \{i \,|\, 1 \le i \le n,\, (\mathbf{v},\mathbf{b}_i) < 0\}\,, \\ Z &= \{i \,|\, 1 \le i \le n,\, (\mathbf{v},\mathbf{b}_i) = 0\}\,. \end{aligned}$$

Then, if we define Λ by

$$\Lambda = \sum_{i\in P} t_i(\mathbf{v},\mathbf{b}_i) > 0\ ,$$

we also have

$$\Lambda = \sum_{i\in M} t_i(-(\mathbf{v},\mathbf{b}_i))\ .$$

With this, we can rewrite our expression for $\mathbf{x}$ as

$$\begin{aligned} \mathbf{x} &= \sum_i s_i\mathbf{a}_i + \sum_{i\in Z} t_i\mathbf{b}_i + \sum_{i\in P} t_i\mathbf{b}_i + \sum_{j\in M} t_j\mathbf{b}_j \\ &= \sum_i s_i\mathbf{a}_i + \sum_{i\in Z} t_i\mathbf{b}_i + \frac{1}{\Lambda}\sum_{i\in P}\left(\sum_{j\in M} t_j(-(\mathbf{v},\mathbf{b}_j))\right) t_i\mathbf{b}_i \\ &\qquad + \frac{1}{\Lambda}\sum_{j\in M}\left(\sum_{i\in P} t_i(\mathbf{v},\mathbf{b}_i)\right) t_j\mathbf{b}_j \\ &= \sum_i s_i\mathbf{a}_i + \sum_{i\in Z} t_i\mathbf{b}_i + \sum_{i\in P}\left(\sum_{j\in M}\frac{t_it_j}{\Lambda}((-(\mathbf{v},\mathbf{b}_j))\mathbf{b}_i + (\mathbf{v},\mathbf{b}_i)\mathbf{b}_j)\right)\ . \end{aligned}$$

We see that the expression on the last line belongs to $\mathrm{Hull}(\mathsf{A},\, FM(\mathsf{B},\mathbf{v}))$. $\square$

To make the technicalities easier, we consider a special case of a subspace or quotient space. Specifically, we start with a convex cone K in $\mathbb{R}^{d+1}$, and recall the linear map ι_0. We also define

$$\pi_0 : \mathbb{R}^{d+1} \to \mathbb{R}^d : (x_1,\ldots,x_d,x_{d+1})^T \mapsto (x_1,\ldots,x_d)^T\ . \tag{4.1}$$

Note that

$$(\iota_0\mathbf{x},\mathbf{y}) = (\mathbf{x},\pi_0\mathbf{y}), \quad \mathbf{x}\in\mathbb{R}^d\,, \quad \mathbf{y}\in\mathbb{R}^{d+1}\,. \tag{4.2}$$

We will also describe our convex cones withut using line generators or hyperplane constraints. As shown by Proposition 2.5, this entails no loss of generality.

Now, for a general convex cone $K \subseteq \mathbb{R}^{d+1}$, we define

$$K^\iota = \{\mathbf{x}\in\mathbb{R}^d \mid \iota_0\mathbf{x}\in K\} = \pi_0(K\cap\mathbf{e}_{d+1}^\perp)$$

and

$$K^\pi = \{\pi_0\mathbf{y} \mid \mathbf{y}\in K\}\,.$$

We wish to prove that, if $\mathsf{B}\in\mathbb{R}^{(d+1)\times n}$, and $K = \mathrm{Hull}(\mathsf{O},\mathsf{B})$, then $K^\iota = \mathrm{Hull}(\mathsf{O},\mathsf{B}')$ for some $\mathsf{B}'\in\mathbb{R}^{d\times n'}$. Similarly, we wish to prove that if $K = \mathrm{Poly}(\mathsf{O},\mathbf{B})$, then $K^\pi = \mathrm{Poly}(\mathsf{O},\mathsf{B}')$.

Proposition 4.3. *Let* $\mathsf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ *be a list of vectors in* $\mathbb{R}^d$*, and let* $K = \mathrm{Hull}(\mathsf{O}, \mathsf{B})$*. Let* B' *be the list of vectors* $\pi_0 \mathbf{b}'$ *where* $\mathbf{b}'$ *is an element of* $FM(\mathsf{B}, \mathbf{e}_{d+1})$*. Then*

$$K^\pi = \mathrm{Hull}(\mathsf{O}, \mathsf{B}') \,.$$

Proof. It follows directly from Proposition 4.2 that

$$\begin{aligned} K^\iota &= \pi_0(K \cap \mathbf{e}_{d+1}^\perp) \\ &= \pi_0(\mathrm{Hull}(\mathsf{O}, FM(\mathsf{B}, \mathbf{e}_{d+1}))) \\ &= \mathrm{Hull}(\mathsf{O}, \mathsf{B}') \,. \end{aligned}$$

□

Proposition 4.4. *Let* K *be a convex cone in* $\mathbb{R}^{d+1}$*. Then*

$$K^{\pi\circ} = K^{\circ\iota} \,.$$

Proof. Let $\mathbf{x} \in \mathbb{R}^d$. Then the statement that $\mathbf{x} \in K^{\pi\circ}$ is equivalent to the statement that for all $\mathbf{y} \in K$, $(\mathbf{x}, \pi_0 \mathbf{y}) \leq 0$. This in turn is equivalent to the statement that for all $\mathbf{y} \in K$, $(\iota_0 \mathbf{x}, \mathbf{y}) \leq 0$. But the latter statement is equivalent to the statement that $\iota_0 \mathbf{x} \in K^\circ$, that is, to $\mathbf{x} \in K^{\circ\iota}$. □

Proposition 4.5. *Let* K *be an H-cone in* $\mathbb{R}^{d+1}$*:*

$$K = \mathrm{Poly}(\mathsf{O}, \mathsf{B}),\ \mathsf{B} \in \mathbb{R}^{(d+1)\times n} \,.$$

Then

$$K^\pi = \mathrm{Poly}(\mathsf{O}, \mathsf{B}')$$

where B' *is defined as in Proposition 4.3.*

Proof. By Theorem 3.3,

$$K^\circ = \mathrm{Hull}(\mathsf{O}, \mathsf{B}) \,.$$

By Proposition 4.3,

$$K^{\circ\iota} = \mathrm{Hull}(\mathsf{O}, \mathsf{B}') \,.$$

By Proposition 4.4,

$$K^{\pi\circ} = \mathrm{Hull}(\mathsf{O}, \mathsf{B}') \,.$$

By Proposition 3.2,

$$K^\pi = \mathrm{Poly}(\mathsf{O}, \mathsf{B}') \,.$$

□

It is clear that we can express the intersection of $\mathrm{Hull}(\mathsf{O}, \mathsf{B})$ with any subspace as a conical hull, and the projection of $\mathrm{Poly}(\mathsf{O}, \mathsf{B})$ on any quotient space as an H-cone, by linear transformations and repeated application of Fourier-Motzkin elimination. This is not a good way to go, computationally, because the number of generators of halfplanes may grow horribly fast. If B contains n vectors, then the number of vectors in $FM(\mathsf{B}, \mathbf{v})$ may be as large as $n^2/4$. Many of these may be redundant; we shall see later how to discard the redundant vectors or even avoid computing them in the first place. (We shall also find a less inelegant way of treating line generators and hyperplane constraints, than sweeping them under the rug into the lists of rays or halfspaces.)

5. Fundamental theorems

We can now prove the main theorem, in three different forms. First we consider convex cones; then, convex polyhedra which may not be bounded, finally, bounded convex polyhedra or **polytopes**.

Theorem 5.1. *(Main theorem for cones) A subset of $\mathbf{R}^d$ is a conical hull if and only if it is an H-cone.*

Proof. First, let $K = \mathrm{Hull}(\mathsf{O}, \mathsf{B}) \subseteq \mathbb{R}^d$, where $\mathsf{B} \in \mathbb{R}^{d\times n}$; we must prove that K is an H-cone. Now if $\mathsf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, then

$$\begin{aligned} K &= \{\Sigma t_\beta \mathbf{b}_\beta \,|\, \mathbf{t} \geq \mathbf{0}, \mathbf{t} \in \mathbb{R}^n\} \\ &= \{\mathbf{x} \in \mathbb{R}^d \mid \exists\, \mathbf{t} \in \mathbb{R}^n \,:\, \mathbf{t} \geq \mathbf{0},\ \mathbf{x} = \Sigma t_\beta \mathbf{b}_\beta\}\,. \end{aligned}$$

The set

$$\{\begin{pmatrix}\mathbf{x}\\ \mathbf{t}\end{pmatrix} \in \mathbb{R}^{d+n} \mid \mathbf{t} \geq \mathbf{0},\ \mathbf{x} = \Sigma t_\beta \mathbf{b}_\beta\}$$

is clearly an H-cone, and K is obtained from it by projecting out the last n components. Therefore the conclusion that K itself is an H-cone follows from n applications of Proposition 4.5.

Second, let $C = \mathrm{Poly}(\mathsf{O}, \mathsf{B})$; we must prove that C is a conical hull. By Proposition 3.2, $C^\circ = \mathrm{Hull}(\mathsf{O}, \mathsf{B})$. But now, by the first part of this theorem, $C^\circ = \mathrm{Poly}(\mathsf{O}, \mathsf{C})$. Therefore, by Theorem 3.3, $C = C^{\circ\circ} = \mathrm{Hull}(\mathsf{O}, \mathsf{C})$. □

Theorem 5.2. *(Main theorem for unbounded polyhedra) A subset of $\mathbb{R}^d$ is a convex-conical hull if and only if it is an H-polyhedron.*

Proof. First, let $K = \mathrm{Hull}(\mathsf{A}, \mathsf{B}, \mathsf{C})$, where A, B, and C are lists of vectors in $\mathbb{R}^d$; we must prove that K is an H-polyhedron. By Proposition 2.7, there is a conical hull $\hat{K} = \mathrm{Hull}(\hat{\mathsf{A}}, \hat{\mathsf{B}})$ in R^{d+1} such that $\iota K = \hat{K} \cap \iota\mathbb{R}^d$. By Theorem 5.1, $\hat{K} = \mathrm{Poly}(\mathsf{O}, \mathsf{C})$ for some $\mathsf{C} \in \mathbb{R}^{(d+1)\times n}$. That is,

$$\iota K = \mathrm{Poly}(\mathsf{O}, \mathsf{C}) \cap \iota\mathbb{R}^d\,.$$

From Proposition 2.2, we see that

$$K = \mathrm{Poly}\begin{pmatrix}\mathsf{O} & \mathsf{D}\\ 0 & \mathsf{w}\end{pmatrix}$$

for suitable D and w.

Second, let

$$K = \mathrm{Poly}\begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{v} & \mathsf{w}\end{pmatrix}\ ;$$

we must prove that K is a convex-conical hull. By Proposition 2.2,

$$\iota K = \mathrm{Poly}(\hat{\mathsf{A}}, \hat{\mathsf{B}}) \cap \mathbb{R}^d\ ;$$

and by Theorem 5.1, $\mathrm{Poly}(\hat{\mathsf{A}}, \hat{\mathsf{B}}) = \mathrm{Hull}(\mathsf{O}, \mathsf{C})$ for a suitable list C of vectors in $\mathbb{R}^{d+1}$. Now the form of C given in Proposition 2.2 ensures that each vector $\mathbf{c}$ in C satisfies $c_{d+1} \geq 0$. We populate two lists D and E of vectors in $\mathbb{R}^d$ as follows: for every $\mathbf{c} = (c_1, \ldots, c_d, c_{d+1})^T$ in the list C, if $c_{d+1} = 0$ then let $(c_1, \ldots, c_d)^T$ belong to D; and if $c_{d+1} > 0$ then let $(c_1/c_{d+1}\,, \ldots, c_d/c_{d+1})^T$ belong to E. Then $K = \mathrm{Hull}(\mathsf{O}, \mathsf{D}, \mathsf{E})$. □

A bounded convex polyhedron is also called a **convex polytope**.

Theorem 5.3. *(Main theorem for polytopes) A subset of $\mathbb{R}^d$ is the convex hull of a finite set of vectors in $\mathbb{R}^d$ if and only if it is a bounded H-polyhedron.*

Proof. Let P be the convex hull of a finite set V of vectors in $\mathbb{R}^d$. Then, by Theorem 5.2, P is an H-polyhedron, and it clearly is bounded.

Conversely, suppose that P is a bounded H-polyhedron. Then, again by Theorem 5.2, P is a convex-conical hull $\mathrm{Hull}(\mathsf{O}, \mathsf{D}, \mathsf{E})$. But P is bounded, so in this expression for P the set D must be empty. □

6. Descriptions

To describe algorithms for dealing with convex cones, it is useful to be somewhat wordy about how the cones themselves are described. We use the word **descriptor** for a vector in $\mathbb{R}^d$.

As we saw in §5, we can describe a particular convex cone by generators, or by constraints. We may define **two-sided** and **one-sided** descriptors for either kind of description..

In the description of K by generators, each two-sided descriptor spans a **line** that is contained in K, and each one-sided descriptor spans a **ray** in K.

In the description of K by constraints, each two-sided descriptor $\mathbf{v}$ determines a **hyperplane** $\{\mathbf{x} \in \mathbb{R}^d | (\mathbf{v}, \mathbf{x}) = 0\}$, and each one-sided descriptor $\mathbf{w}$ determines a **halfspace** $\{\mathbf{x} \in \mathbb{R}^d | (\mathbf{w}, \mathbf{x}) \leq 0\}$.

Table 1 summarizes the terms I have just introduced:

	Two-sided	One-sided
Generators	lines	rays
Constraints	hyperplanes	halfspaces

Table 1. Nomenclature for descriptors

The **double description** of K consists of the two kinds of description. We shall represent a double description thus:

$$\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D})) \tag{6.1}$$

where A is the list of line generators, B is the list of ray generators, C is the list of hyperplane constraints, and D is the list of halfspace constraints.

Much of the "fundamental theory" of convex cones can be summarized by two remarks:

(1) the cone described by the above $\mathcal{D}$ is
$$K = \mathrm{Hull}(\mathsf{A}, \mathsf{B}) = \mathrm{Poly}(\mathsf{C}, \mathsf{D})\ ;$$
(2) a double description of K° is obtained from that of K by exchanging the generators with the constraints.

Every line descriptor is necessarily orthogonal to all of the constraint descriptors, and similarly every hyperplane descriptor is orthogonal to all of the generators. The inner product of a ray descriptor with halfspace generator may be 0 or strictly negative. Table 2 summarizes the inner products of the generator and constraint descriptors.

		Generator	
		line	ray
Constraint	hyperplane	0	0
	halfplane	0	≤ 0

TABLE 2. Inner products of descriptors

6.1. **The intuitive notion of "best" description.** Evidently there can be many different double descriptions of the same cone. We would like to find the best descriptions, even if we are not quite sure what "best" means here. What are we looking for? The best descriptions should be economized, not using unnecessary descriptors. And it would be nice if the best descriptions made certain intrinsic properties of the cones apparent "on inspection."

Two intrinsic properties of any convex polyhedron are these:

Definition 6.1. Let P be a convex set in $\mathbb{R}^d$. Then the **lineality space** of P, denoted $\mathrm{lineal}(P)$, is

$$\mathrm{lineal}(P) := \{\mathbf{y} \in \mathbb{R}^d \,|\, \mathbf{x} + t\mathbf{y} \in P \;\forall \mathbf{x} \in P, t \in \mathbb{R}\} \;.$$

The **affine hull** of P, denoted $\mathrm{aff}(P)$, is the smallest affine subspace containing P.

Note that if P contains the origin (for instance, if P is a cone), then $\mathrm{aff}(P)$ is a vector space.

If K is a closed convex polyhedral cone and $\mathrm{lineal}(K) = \{\mathbf{0}\}$, then we say that K is **pointed**. If $\mathrm{aff}(K) \;= \mathbb{R}^d$, then we say that K is d-dimensional or **of full dimension**; then K has interior points.

It turns out that $\mathrm{lineal}(K)$ and $\mathrm{aff}(K)$ can both be revealed by a properly composed description. We can make sure that, in (6.1), A is a basis of $\mathrm{lineal}(K)$ and C is a basis of $(\mathrm{aff}(K))^{\perp}$. In particular,

$$\dim \mathrm{lineal}(K) = \#(\mathsf{A})$$

and

$$\dim \mathrm{aff}(K) = d - \#(\mathsf{C}) \;.$$

For the one-sided descriptors, the important thing is not to be redundant. Informally, a descriptor is redundant if it can be removed from the description without having any effect on the thing being described. We will give this notion a more rigorous definition. We also demonstrate some criteria for deciding when a descriptor is redundant.

It is convenient to define the optimization of double descriptions in stages.

6.2. **First-Stage Optimization.**

Definition 6.2. Let K be a convex polyhedral cone, and lst $\mathcal{D}$ be a double description of K. A one-sided descriptor $\mathbf{v}$ is **saturated** if it is orthogonal to all the one-sided descriptors in the opposite description.

Proposition 6.3. *Let K be a convex polyhedral cone in $\mathbb{R}^d$, with a double description $\mathcal{D}$; let $\mathbf{v}$ be a saturated one-sided descriptor. If $\mathbf{v}$ is a ray descriptor, then the line $R_{\mathbf{v}}$ is contained in K. If $\mathbf{v}$ is a halfspace descriptor, then the hyperplane $\{\mathbf{x} \in \mathbb{R}^d \,|\, (\mathbf{x}, \mathbf{v}) = 0\}$ contains K.*

Proof. If $\mathbf{v}$ is a saturated ray descriptor, then $R_{\mathbf{v}}$ satisfies all the constraints in $\mathcal{D}$. On the other hand, if $\mathbf{v}$ is a saturated halfspace descriptor, then all the generators in $\mathcal{D}$ satisfy the hyperplane constraint. □

Corollary 6.4. *Let K be a convex polyhedral cone in $\mathbb{R}^d$, with a double description $\mathcal{D}$. Then another double description of K may be obtained by moving each saturated ray descriptor into the list of line descriptors, and each saturated halfspace descriptor into the list of hyperplane descriptors.*

Definition 6.5. Let K be a convex polyhedral cone in $\mathbb{R}^d$, with a double description $\mathcal{D}$. Then $\mathcal{D}$ is **first-stage optimized** if it contains no saturated one-sided descriptors.

The corollary shows that every convex polyhedral cone does have a first-stage optimized double description.

Proposition 6.6. *Let K be a convex polyhedral cone in $\mathbb{R}^d$, with a double description $\mathcal{D}$ which is first-stage optimized. Then* $\mathrm{lineal}(K)$ *is spanned by the line generators of $\mathcal{D}$.*

Proof. From the definition,

$$\mathrm{lineal}(K) = \{\mathbf{x} \in K \mid R_{\mathbf{x}} \subseteq K\} \ .$$

Let $\mathcal{D}$ be as in (6.1). It is evident that every element of A is contained in $\mathrm{lineal}(K)$. Conversely, if $\mathbf{x} \in \mathrm{lineal}(K)$, then we will show that $\mathbf{x}$ is a linear combination of the elements of A. Because $\mathbf{x} \in K$, we have

$$\mathbf{x} = \Sigma t_i \mathbf{a}_i + \Sigma u_i \mathbf{b}_i \ ,$$

where the $\mathbf{a}_i$ are elements of $\mathcal{A}$, the $\mathbf{b}_i$ are elements of $\mathcal{B}$, and $u_i \geq 0$. Now suppose $u_i > 0$ for some i. By hypothesis, $\mathbf{b}_i$ is not saturated; therefore there is a halfspace constraint descriptor $\mathbf{d}_j$ such that $(\mathbf{b}_i, \mathbf{d}_j) < 0$. From this it follows that $(\mathbf{x}, \mathbf{d}_j) < 0$, and therefore, in particular, $-\mathbf{x} \notin K$. This implication shows that if $R_{\mathbf{x}} \subseteq K$, then all the terms $u_i \mathbf{b}_i$ in the above expression for $\mathbf{x}$ must vanish, q.e.d. □

Proposition 6.7. *Let K be a convex polyhedral cone in $\mathbb{R}^d$, with a double description $\mathcal{D}$ which is first-stage optimized. Then* $(\mathrm{aff}(K))^{\perp}$ *is spanned by the hyperplane constraint descriptors of $\mathcal{D}$.*

Proof. If $\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D}))$, then $\mathcal{D}^\circ = ((\mathsf{C}, \mathsf{D}), (\mathsf{A}, \mathsf{B}))$ is a double descriptor of K°. Now by hypothesis, $R_{\mathbf{x}} \subseteq K^\circ$; so, by Proposition 6.6, $\mathbf{x}$ is in the vector space spanned by the vectors in D. □

6.3. **Second-stage optimization.**

Definition 6.8. Let K be a convex polyhedral cone in $\mathbb{R}^d$, with a double description $\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D}))$. Then $\mathcal{D}$ is **second-stage optimized** if it is first-stage optimized and, in addition, each of A and C is a linearly independent list of vectors.

Gaussian elimination is a well-known method for finding, from a finite set of vectors, another, linearly independent set which generates the same vector space. The method can be described as a sequence of transformations of a list of vectors. It proceeds through the list, taking as "current vector $\mathbf{v}$" the first element, then the second, and so on until the list is exhausted. If the current vector is $\mathbf{0}$, then it is eliminated. Otherwise, let ι be the index of a non-zero component v_ι of $\mathbf{v}$. It may be felt convenient, but it is not strictly necessary, to scale $\mathbf{v}$ so that $v_\iota = 1$. Then, every other vector $\mathbf{w}$ is altered, by subtracting from it a multiple of $\mathbf{v}$ so that w_ι becomes 0. This step having been performed for $\mathbf{v}$, the process goes on to the next vector in the list.

Each step in the process of Gaussian elimination leads to a list of vectors which spans the same vector space. Thus the vector space is the same at the end of the process as it was at the beginning. At the end of the process, for each vector that remains in the list there is a component index, such that this vector, and only this vector, has a non-zero component at this index. From this property of the list, it easily follows that the list of vectors is linearly independent.

6.4. **Third-stage optimization.**

Definition 6.9. Let K be a closed convex polyhedral cone, and let $\mathcal{L} = \mathrm{lineal}(K)$. Let $\mathbf{v}, \mathbf{w} \in K \setminus \mathcal{L}$. Then $\mathbf{v}$ and $\mathbf{w}$ are **lineally equivalent** (for K) if $\mathbf{w} = \lambda\mathbf{v} + \mathbf{u}$ for some non-zero λ and some $\mathbf{u} \in \mathcal{L}$.

Lineal equivalence is evidently an equivalence relation. (You would hope so, would you not?) In the generator description of a cone, ray descriptors can be replaced by others which are lineally equivalent, without changing the cone being described. To put this more formally, let K be a closed convex polyhedral cone, described by generators:

$$K = \mathrm{Hull}(\mathsf{A}, \mathsf{B}) \ ;$$

suppose that A is a basis of $\mathrm{lineal}(K)$ — as will be the case, if (A, B) is part of a double description that is second-stage optimized. Then the cone is not changed if any element of B is replaced by a lineally equivalent vector. Furthermore, it is clear that if two elements of B are lineally equivalent, then one of them may be removed from the list without changing the cone.

Lineal equivalence also applies to the constraint description of a cone:

$$K = \mathrm{Poly}(\mathsf{C}, \mathsf{D}) \ ;$$

by analogy, we suppose that C is a basis of $(\mathrm{aff}\, K)^\perp = \mathrm{lineal}(K^\circ)$. Then a halfspace descriptor $\mathbf{d}$ is lineally equivalent (for K°) to a vector obtained from it by non-zero scaling and addition of a vector from $\mathrm{lineal}(K^\circ)$.

Definition 6.10. Let K be a closed convex polyhedral cone, and let

$$\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D}))$$

be a double description of K. Thn $\mathcal{D}$ is **third-stage optimized** if it is second-stage optimized and moreover no two elements of B are lineally equivalent (for K) and no two elements of D are lineally equivalent (for K°).

Gaussian elimination can be extended so as to facilitate the detection of lineally equivalent descriptors. For instance, in the generator description (A, B), the "current vector" $\mathbf{v}$ ranges over the elements of A; but the vector $\mathbf{w}$ which is altered to make $w_\iota = 0$ ranges over both A and B. When this has been done, two vectors in the transformed B are lineally equivalent if and only if they are proportional.

7. Faces, descriptors, and redundancy

7.1. Faces.

Definition 7.1. Let P be a closed convex polyhedron in $\mathbb{R}^d$. Let $\mathbf{c} \in \mathbb{R}^d$ and $c_0 \in \mathbb{R}$. Then $\mathbf{c}$ and c_0 **define a valid inequality for** P if $(\mathbf{c}, \mathbf{x}) \leq c_0$ for every $\mathbf{x} \in P$.

Definition 7.2. Let P be a closed convex polyhedron in $\mathbb{R}^d$. Then a **face** of P is any set of the form

$$P \cap \{\mathbf{x} \in \mathbb{R}^d \,|\, (\mathbf{c}, \mathbf{x}) = c_0\}$$

where $\mathbf{c}$ and c_0 define a valid inequality for P.

One sees that $\emptyset$ and P are faces of P. Any non-empty face of P is a closed convex polyhedron. It is common to use the terms **vertex**, **edge**, and **facet** for faces whose dimension is, respectively, 0, 1, and $\dim(P) - 1$.

Proposition 7.3. *Let K be a convex polyhedral cone in $\mathbb{R}^d$, and let F be a non-empty face of K. Then F is the intersection of K with a hyperplane passing through the origin.*

Proof. By hypothesis, there exist $\mathbf{c}$ and c_0 such that

$$F = K \cap \{\mathbf{x} \in \mathbb{R}^d \,|\, (\mathbf{c}, \mathbf{x}) = c_0\}$$

and

$$(\mathbf{c}, \mathbf{x}) \leq c_0 \quad \forall \mathbf{x} \in K \ .$$

Now let $\mathbf{x}_0 \in F$; then $(\mathbf{c}, \mathbf{x}_0) = c_0$. However, $\lambda \mathbf{x}_0 \in K$ for all nonnegative λ, which implies

$$\lambda c_0 \leq c_0 \quad \forall \lambda \geq 0 \ .$$

It is easy to see from this that $c_0 = 0$. □

Proposition 7.4. *If K is a convex polyhedral cone and F is a non-empty face of K, then $F \supseteq \operatorname{lineal}(K)$.*

Proof. By hypothesis and Proposition 7.3, there exists $\mathbf{c}$ such that $(\mathbf{c}, \mathbf{x}) \leq 0$ for all $\mathbf{x} \in K$ and also $F = \{\mathbf{x} \in K \,|\, (\mathbf{c}, \mathbf{x}) = 0\}$. Now if $\mathbf{x} \in \operatorname{lineal}(K)$, then $\mathbf{x} \in K$ and $-\mathbf{x} \in K$, from which it follows that $(\mathbf{c}, \mathbf{x}) = 0$. □

Proposition 7.5. *Let K be a closed convex cone; then* $\operatorname{lineal}(K)$ *is a face of K.*

Proof. Let $((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D}))$ be a double description of K which is second-stage optimized; let $\mathbf{f}$ be the sum of the vectors in D. Then $\mathbf{f}$ and 0 define a valid inequality for K; we claim that the face of K defined by $(\mathbf{f}, \mathbf{x}) = 0$ is $\operatorname{lineal}(K)$.

Indeed, $(\mathbf{f}, \mathbf{a}) = 0$ for every $\mathbf{a}$ belonging to A. If $\mathbf{b}$ belongs to B, then because $\mathbf{b}$ is not saturated $(\mathbf{f}, \mathbf{b}) < 0$. Now, if $\mathbf{x} \in K$ then

$$\mathbf{x} = \Sigma t_\alpha \mathbf{a}_\alpha + \Sigma u_\beta \mathbf{b}_\beta, \quad u_\beta \geq 0 \ .$$

In order to have $(\mathbf{f}, \mathbf{x}) = 0$, it must be the case that $u_\beta = 0 \ \forall \beta$. This $\mathbf{x}$ is in the vector space generated by A. which is just $\operatorname{lineal}(K)$. □

7.2. **Descriptors and redundancy.** We need to formalize the concept of redundancy for one-sided descriptors. Informally, a descriptor is redundant if dropping it from the description does not change the cone. So it would seem that being redundant is essentially dependent, not just on the descriptor itself but also on the whole description. The following proposition will be used to show that this is not quite true.

Proposition 7.6. *Let K be a closed convex polyhedral cone,*

$$K = \mathrm{Hull}(\mathsf{A}, \mathsf{B}) \ ,$$

where A *is a basis of* $\mathrm{lineal}(K)$ *and none of the elements of* B *belong to* $\mathrm{lineal}(K)$*. Let* $\mathcal{L} = \mathrm{lineal}(K)$*. Let* $\mathbf{b}$ *be an element of* B*, and let* B' *be the result of removing* $\mathbf{b}$ *from* B*. If*

$$K' = \mathrm{Hull}(\mathsf{A}, \mathsf{B}') \neq K \ ,$$

then $K \cap (\mathcal{L} \oplus L_{\mathbf{b}})$ *is a face of* K*.*

Proof. If b were a member of K', then K would equal K'; so we must have $\mathsf{b} \notin K'$. Therefore, there is a vector $\mathbf{f} \in (K')^\circ$ such that $(\mathbf{f}, \mathbf{b}) > 0$. On the other hand, because $\mathcal{L}$ is a face of K, there is a vector $\mathbf{g} \in \mathbf{R}^d$ such that $(\mathbf{g}, \mathbf{x}) = 0$ if $\mathbf{x} \in \mathcal{L}$, and

$$(\mathbf{g}, \mathbf{b}) < 0, \ (\mathbf{g}, \mathbf{b}') < 0, \ \forall \mathbf{b}' \in \mathsf{B}' \ .$$

Let $\rho = (\mathbf{f}, \mathbf{b})/(\mathbf{g}, \mathbf{b}) < 0$, and let $\mathbf{h} = \mathbf{f} - \rho \mathbf{g}$. Then $(\mathbf{h}, \mathbf{x}) = 0$ if $\mathbf{x} \in \mathcal{L}$, and

$$(\mathbf{h}, \mathbf{b}) = 0, \ (\mathbf{h}, \mathbf{b}') < 0 \ \forall \mathbf{b}' \in \mathsf{B}' \ .$$

It follows that

$$K \cap (\mathcal{L} \oplus L_{\mathbf{b}}) = \{\mathbf{x} \in K \,|\, (\mathbf{h}, \mathbf{x}) = 0\} \ .$$

□

Theorem 7.7. *Let K be a convex polyhedral cone in $\mathbb{R}^d$; let $\mathcal{L} = \mathrm{lineal}(K)$; and let $\mathbf{v} \in K \setminus \mathcal{L}$. Then the following three statements are equivalent:*

- (i) *$K \cap (\mathcal{L} \oplus L_{\mathbf{v}})$ is a face of K;*
- (ii) *in any expression of $\mathbf{v}$ as a sum of elements of K, all the terms must belong to $\mathcal{L} \oplus L_{\mathbf{v}}$;*
- (iii) *in any description of K by generators, the ray generators must include an element of $K \cap (\mathcal{L} \oplus L_{\mathbf{v}})$.*

Proof. These implications have to be checked:

(i)⇒ (ii):

Suppose that $K \cap (\mathcal{L} \oplus L_{\mathbf{v}})$ is a face of K. Then there is a vector $\mathbf{f} \in \mathbb{R}^d$ such that

$$\mathbf{x} \in K \Rightarrow (\mathbf{f}, \mathbf{x}) \leq 0 \ ;$$

$$K \cap (\mathcal{L} \oplus L_{\mathbf{v}}) = \{\mathbf{x} \in K \,|\, (\mathbf{f}, \mathbf{x}) = 0\} \ .$$

Thus, if $\mathbf{v} = \mathbf{w}_1 + \cdots + \mathbf{w}_n$ where $\mathbf{w}_i \in K$, then we have $(\mathbf{f}, \mathbf{w}_i) \leq 0$ for $i = 1, \ldots, n$ and also $(\mathbf{f}, \mathbf{v}) = 0$. The last equality forces $(\mathbf{f}, \mathbf{w}_i) = 0$, which implies $\mathbf{w}_i \in K \cup (\mathcal{L} \oplus L_{\mathbf{v}})$.

(ii)⇒ (iii):

Suppose that condition (ii) is satisfied. Let $K = \mathrm{Hull}(\mathsf{A}, \mathsf{B})$. Then

$$\mathbf{v} = \Sigma t_\alpha \mathbf{a}_\alpha + \Sigma u_\beta \mathbf{b}_\beta \ ;$$

since $\mathbf{v} \notin \mathcal{L}$, one of the terms $u_\beta \mathbf{b}_\beta$ must be non-zero. By supposition, every term in this sum must be a member of $\mathcal{L} \oplus L_\mathbf{v}$; thus this particular $\mathbf{b}_\beta$ must belong to $K \cap (\mathcal{L} \oplus L_\mathbf{v})$.

(iii)⇒ (i):
Suppose that condition (iii) is satisfied. Let $K = \mathrm{Hull}(\mathsf{A}, \mathsf{B})$; we may assume that A is a basis of $\mathcal{L}$ and no element of B belongs to $\mathcal{L}$. By supposition, B must contain at least one element of $K \cap (\mathcal{L} \oplus L_\mathbf{v})$. The convex-conical hull will not be changed if one such element is replaced by $\mathbf{v}$ itself; nor will it be changed if other elements of $K \cap (\mathcal{L} \oplus L_\mathbf{v})$ are omitted. Thus, we may assume that

$$\mathsf{B} = (\mathbf{v} = \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n),$$

where $\mathbf{b}_2, \ldots, \mathbf{b}_n$ do not belong to $K \cap (\mathcal{L} \oplus L_\mathbf{v})$.

Let

$$\mathsf{B}' = (\mathbf{b}_2, \ldots, \mathbf{b}_n) \ ,$$

and

$$K' = \mathrm{Hull}(\mathsf{A}, \mathsf{B}') \ .$$

By condition (iii), $K' \neq K$. Proposition 7.6 now implies the desired conclusion. □

Definition 7.8. Let K be a closed convex polyhedral cone. Then a vector $\mathbf{v}$ is **relatively extremal for** K if it satisfies the conditions of Proposition 7.7.

Definition 7.9. Let K be a closed convex polyhedral cone, with a double description

$$\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D})) \ .$$

Then an element of B is a **redundant generator for** $\mathcal{D}$ if it is not relatively extremal for K; and an element of D is a **redundant constraint for** $\mathcal{D}$ if it is not relatively extremal for K°.

Proposition 7.10. *Let K be a closed convex polyhedral cone, with a double description*

$$\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D})) \ .$$

Let $\mathbf{b}$ *be an element of* B *that is redundant for* $\mathcal{D}$*; let* B' *be the result of removing* $\mathbf{b}$ *from* B*. Then* $\mathrm{Hull}(\mathsf{A}, \mathsf{B}') = K$.

Proof. By hypothesis, $\mathbf{b}$ is not relatively extremal for K. By Definition 7.8, $K \cap (\mathrm{lineal}(K) \oplus L_\mathbf{b})$ is not a face of K. The conclusion follows from Proposition 7.6. □

Corollary 7.11. *Let K be a closed convex polyhedral cone, with a double description*

$$\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D})) \ .$$

Let $\mathbf{d}$ *be an element of* D *that is redundant for* $\mathcal{D}$*; let* D' *be the result of removing* $\mathbf{d}$ *from* D*. Then* $\mathrm{Poly}(\mathsf{C}, \mathsf{D}') = K$.

Proof. By hypothesis, $\mathbf{d}$ is not relatively extremal for K°. Therefore, by Proposition 7.10, $K^\circ = \mathrm{Hull}(\mathsf{C}, \mathsf{D}')$. By Theorem 3.3, $K = \mathrm{Poly}(\mathsf{C}, \mathsf{D}')$. □

Definition 7.12. Let K be a closed convex polyhedral cone. Then a double description $\mathcal{D}$ of K is **fully optimized** if it is third-stage optimized and, in addition, no one-sided generators are redundant for $\mathcal{D}$).

It is clear that every closed convex polyhedral cone K has a fully optimized double description

$$\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D})) \ .$$

Its descriptors are related to the intrinsic structure of K:

- A is a basis of $\mathrm{lineal}(K)$.
- B consists of relatively extremal vectors for K. In any other fully optimized double description of K, the ray generators would each be lineally equivalent (for K) to one element of B.
- C is a basis of $\mathrm{lineal}(K^\circ)$, which is the same as $(\mathrm{aff}(K))^\perp$.
- D consists of relatively extremal vectors for K°. In any other fully optimized double description of K, the halfspace constraints would each be lineally equivalent (for K°) to one element of D.

Proposition 7.13. *Let K be a closed convex polyhedral cone in $\mathbb{R}^d$, and let $\mathbf{v}$ be a relatively extremal vector for K. Let $K_1 = K + L_{\mathbf{v}}$. Then*

$$(K_1)^\circ = \{\mathbf{x} \in K^\circ \mid (\mathbf{x}, \mathbf{v}) = 0\} \ ,$$

and $(K_1)^\circ$ is a facet of K°.

Proof. The characterization of $(K_1)^\circ$ is easily checked. Note that because of it, $(K_1)^\circ$ is a face of K°. We need to verify that it has the desired dimension.

Consider a fully optimized double description of K as above. If A_1 is A with $\mathbf{v}$ appended, and B_1 is B with $\mathbf{v}$ removed, then $K_1 = \mathrm{Hull}(\mathsf{A}_1, \mathsf{B}_1)$. Therefore

$$\dim \mathrm{aff}((K_1)^\circ) = d - \dim \mathrm{lineal}(K_1) = d - \dim \mathrm{lineal}(K) - 1 = \dim \mathrm{aff}(K^\circ) - 1 \ ,$$

q.e.d. $\square$

We will use the notation $K^{\circ\mathbf{v}}$ for the facet $(K_1)^\circ$ in this proposition.

8. Detection of Redundant Descriptors

Useful criteria for detecting redundant descriptors can be found using only the inner products of ray and halfspace descriptors; in fact, all that matters is which of these inner products are 0.

By the theory of polarity, there is complete symmetry between the ray descriptors on the one hand and the halfspace descriptors on the other. We will describe the redundancy criteria in terms of eliminating redundant rays.

Definition 8.1. Let

$$\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D}))$$

be a double description of a closed convex cone, with

$$\mathsf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_m) \ ,$$
$$\mathsf{D} = (\mathbf{d}_1, \ldots, \mathbf{d}_n) \ .$$

Then

$$\sigma_i(\mathcal{D}) := \{j \in \{1, \ldots, n\} \mid (\mathbf{b}_i, \mathbf{d}_j) = 0\} \ , \quad i = 1, \ldots, m \ .$$

When the choice of $\mathcal{D}$ is clear from context we shall omit it.

Proposition 8.2. *Let K be a closed convex polyhedral cone, with a double description $\mathcal{D}$ as in Definition 8.1. For any i with $1 \le i \le m$, if*

$$\#(\sigma_i) < \dim \operatorname{aff}(K) - \dim \operatorname{lineal}(K) - 1 ,$$

then the ray descriptor $\mathbf{b}_i$ is redundant for $\mathcal{D}$.

Proof. Suppose, on the contrary, that $\mathbf{b}_i$ is not redundant, but is in fact a relatively extremal vector for K. Then, by Proposition 7.13,

$$\begin{aligned} \dim \operatorname{aff}(K^{\circ \mathbf{b}_i}) &= \dim \operatorname{aff}(K^\circ) - 1 \\ &= d - \dim \operatorname{lineal}(K) - 1 . \end{aligned}$$

But $K^{\circ \mathbf{b}_i}$ is generated by C together with $\{\mathbf{d}_j \mid j \in \sigma_i\}$. Therefore

$$\begin{aligned} \dim \operatorname{aff}(K^{\circ \mathbf{b}_i}) &\le \#\mathsf{C} + \#(\sigma_i) \\ &\le d - \dim \operatorname{aff}(K) + \#(\sigma_i) . \end{aligned}$$

This inequality and the preceding equality together imply that

$$\#(\sigma_i) \ge \dim \operatorname{aff}(K) - \dim \operatorname{lineal}(K) - 1 .$$

□

Corollary 8.3. *Let K be a closed convex polyhedral cone, with a double description $\mathcal{D}$ as in Definition 8.1; suppose also that $\mathcal{D}$ is second-stage optimized. For any i with $1 \le i \le m$, if*

$$\#\sigma_i < d - \#(\mathsf{A}) - \#(\mathsf{C}) - 1 ,$$

then the ray descriptor $\mathbf{b}_i$ is redundant for $\mathcal{D}$.

That proposition gives us a sufficient condition for deciding that a descriptor is redundant. Here is a necessary and sufficient condition, which however takes more work to apply.

Proposition 8.4. *Let K be a closed convex polyhedral cone, and let $\mathcal{D}$ be a double description of K. With the notations of Definition 8.1, a ray descriptor $\mathbf{b}_i$ is redundant for $\mathcal{D}$ if and only if $\sigma_i \subseteq \sigma_j$ for some $j \ne i$.*

Proof. First, let us suppose that $\mathbf{b}_i$ is redundant for $\mathcal{D}$; we must show that $\sigma_i \subseteq \sigma_j$ for some $j \ne i$. Let B' denote a list of all the elements in B except $\mathbf{b}_i$. By Proposition 7.10, $\mathbf{b}_i \in \operatorname{Hull}(\mathsf{A}, \mathsf{B}')$. That is to say,

$$\mathsf{b}_i = \sum_\alpha s_\alpha \mathbf{a}_\alpha + \sum_{j \ne i} t_j \mathbf{b}_j \ , \ t_j \ge 0 \ .$$

There must be an index j such that $t_j > 0$. For this j, and for an element $\mathbf{d}$ of D,

$$(\mathbf{b}_i, \mathbf{d}) = 0 \ \Rightarrow \ (\mathbf{b}_j, \mathbf{d}) = 0 \ ;$$

in other words, $\sigma_i \subseteq \sigma_j$.

Second, let us suppose that $\mathbf{b}_i$ is not redundant for $\mathcal{D}$; we must show that, for every $j \ne i$, $1 \le j \le m$, there is a k, $1 \le k \le n$, such that $k \in \sigma_i$ but $k \notin \sigma_j$. Indeed, by Theorem 7.7(i), $\mathbf{b}_i$ is contained in a face of K that does not contain any of the $\mathbf{b}_j$ with $j \ne i$. Therefore there is a vector $\mathbf{y} \in K^\circ$ such that $(\mathbf{b}_i, \mathbf{y}) = 0$ but $(\mathbf{b}_j, \mathbf{y}) < 0$. Now

$$\mathbf{y} = \sum_\gamma s_\gamma \mathbf{c}_\gamma + \sum_{k=1}^n t_k \mathbf{d}_k \ , \ t_k \ge 0 \ ;$$

there must be a value of k for which $t_k > 0$ and $(\mathbf{b}_j, \mathbf{d}_k) < 0$ but $(\mathbf{b}_i, \mathbf{d}_k) = 0$. That is, $k \notin \sigma_j$ but $k \in \sigma_i$; this is what we needed to show. $\square$

It might happen that two elements of B, say $\mathbf{b}_i$ and $\mathbf{b}_j$, are lineally equivalent. Then $\sigma_i = \sigma_j$, and each one is redundant—given that the other one is present. One may be removed without changing K, but it is not safe to remove both of them. Thus it is better to apply Proposition 8.4 after third-stage optimization.

9. Incremental Operations on Cones and Their Descriptions

By an "incremental operation" we mean adding a single descriptor. This usage sounds right if we are adding a line or a ray, and taking the convex hull of the result. "Adding" a constraint may appear to be quite different; it amounts to taking the intersection of the given cone with a halfspace or hyperplane. In fact, the operations are profoundly similar. If I change K by adding a line or a ray, then I change K° by intersecting it with a hyperplane or a halfspace. In this section, we consider how to obtain a double description of the changed cone, given a double description of the original cone.

The operations $K \mapsto K^\iota$ and $K \mapsto K^\pi$ of §4 are thinly disguised special cases of incremental changes to cones in $\mathbb{R}^{d+1}$. The disguise is pretty transparent for going from K to K^ι, because K^ι is just the pre-image, with respect to ι_0, of $K \cap \iota_0\mathbb{R}^d$. As for K^π, one way of describing the construction is to let K_1 be the convex hull of the union of K with the line generated by $\mathbf{e}_{d+1}$; then K^π is the pre-image of K_1 with respect to ι_0.

The computational details are quite symmetrical between generators and constraints. We will assume that we are adding a line or a ray, and describe how the hyperplane and halfspace constraints are affected.

So, let K be a closed polyhedral cone in $\mathbb{R}^d$, with double description

$$\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D}))\ .$$

The descriptor $\mathbf{v}$ is to be added to either A or B, to define the cone K'. Our problem is to determine how C and D must be changed to obtain a double description of K'. Two cases must be distinguished. In the first case, there is a descriptor $\mathbf{c}$ in C such that $(\mathbf{v}, \mathbf{c}) \neq 0$; in the second case, there is no such $\mathbf{c}$.

9.1. First case.

The geometric significance of the inequality $(\mathbf{v}, \mathbf{c}) \neq 0$ is that, while K lies in the hyperplane $\mathbf{c}^\perp$ orthogonal to $\mathbf{c}$, the line or ray being added lies outside this hyperplane. Thus the affine hull of K' is larger than that of K.

To find a constraint description of K', we begin by observing that an arbitrary multiple of $\mathbf{c}$ can be added to each of the other constraints in (C, D) without having any effect on the cone being described. Let us suppose this to be done, in such a way that, except for $\mathbf{c}$, every constraint in (C, D) is orthogonal to $\mathbf{v}$. We may also assume that $(\mathbf{v}, \mathbf{c}) < 0$.

The geometrical picture of this transformation is that each of the other hyperplanes orthogonal to a constraint descriptor is swung around, pivoting on its intersection with $\mathbf{c}^\perp$, until it passes through the new generating descriptor. If the new descriptor is a line, then K' is described by these constraints, with $\mathbf{c}$ omitted. If the new descriptor is a ray, then $\mathbf{c}$ survives as a halfspace constraint.

The last paragraph *may* be intuitive; but let us also have a proof.

Proposition 9.1. *Let*

$$K = \mathrm{Hull}(\mathsf{A}, \mathsf{B}) \;=\; \mathrm{Poly}(\mathsf{C}, \mathsf{D}) \subseteq \mathbb{R}^d \;.$$

Let $\mathbf{v} \in \mathbb{R}^d$*. Suppose that* $\mathbf{c}$ *is an element of* C *such that* $(\mathbf{v}, \mathbf{c}) < 0$ *but all other elements of* C*, and all elements of* D*, are orthogonal to* $\mathbf{v}$*. Then*

$$\mathrm{Hull}(\mathsf{A} + (\mathbf{v}), \mathsf{B}) = \mathrm{Poly}(\mathsf{C} - (\mathbf{c}), \mathsf{D}) \,, \tag{9.1}$$

$$\mathrm{Hull}(\mathsf{A}, \mathsf{B} + (\mathbf{v})) = \mathrm{Poly}(\mathsf{C} - (\mathbf{c}), \mathsf{D} + (\mathbf{c})) \,. \tag{9.2}$$

Proof. In each equation, the Hull is a subset of the Poly; it is sufficient to check that each generator on the left satisfies each constraint on the right. To prove inclusion the other way, let us suppose that $\mathbf{x}$ is a vector in $\mathbb{R}^d$ which satisfies all the constraints on the right side of the equation; that is, if $\mathbf{c}'$ is an element of $\mathsf{C} - (\mathbf{c})$ then $(\mathbf{x}, \mathbf{c}') = 0$; if $\mathbf{d}$ is an element of D, then $(\mathbf{x}, \mathbf{d}) \leq 0$; and, in the case of equation 9.2, $(\mathbf{x}, \mathbf{c}) \leq 0$. Let

$$\lambda = (\mathbf{x}, \mathbf{c})/(\mathbf{v}, \mathbf{c}) \,.$$

Note that in the case of equation 9.2, $\lambda \geq 0$. Then $\mathbf{x} = \mathbf{x}' + \lambda \mathbf{v}$ where $(\mathbf{x}', \mathbf{c}) = 0$. It can be checked that $\mathbf{x}'$ satisfies all the constraints needed to imply that

$$\mathbf{x}' \in \mathrm{Poly}(\mathsf{C}, \mathsf{D}) = \mathrm{Hull}(\mathsf{A}, \mathsf{B}) \,;$$

it follows that $\mathbf{x}$ belongs to the Hull on the left side of the equation. □

9.2. **Second case.** In the second case, we need to apply Fourier-Motzkin elimination to the halfspace descriptors D. We recall Definition 4.1; after showing that it gives us what we want, we will define a more efficient form of Fourier-Motzkin elimination, and prove that it leads to the same result.

Proposition 9.2. *Let* K *be a closed polyhedral convex cone in* $\mathbb{R}^d$*:*

$$K = \mathrm{Hull}(\mathsf{A}, \mathsf{B}) = \mathrm{Poly}(\mathsf{C}, \mathsf{D}) \;;$$

let $\mathbf{v} \in \mathbb{R}^d$ *be orthogonal to all the elements of* C*. Then*

$$\mathrm{Hull}(\mathsf{A} + (\mathbf{v}),\, \mathsf{B}) = \mathrm{Poly}(\mathsf{C},\, FM(\mathsf{D}, \mathbf{v})) \;. \tag{9.3}$$

If D^- *is a list of those elements* $\mathbf{d}$ *of* D *such that* $(\mathbf{v}, \mathbf{d}) < 0$*, then*

$$\mathrm{Hull}(\mathsf{A},\, \mathsf{B} + (\mathbf{v})) = \mathrm{Poly}(\mathsf{C},\, \mathsf{D}^- + FM(\mathsf{D}, \mathbf{v})) \;. \tag{9.4}$$

Proof. It is more convenient to prove the equivalent statements about the polar cone. Given

$$K^\circ = \mathrm{Hull}(\mathsf{C}, \mathsf{D}) = \mathrm{Poly}(\mathsf{A}, \mathsf{B}) \,,$$

we must prove

$$\mathrm{Hull}(\mathsf{C},\, FM(\mathsf{D}, \mathbf{v})) = \mathrm{Poly}(\mathsf{A} + (\mathbf{v}),\, \mathsf{B}) \tag{9.5}$$

and

$$\mathrm{Hull}(\mathsf{C},\, \mathsf{D}^- + FM(\mathsf{D}, \mathbf{v})) = \mathrm{Poly}(\mathsf{A},\, \mathsf{B} + (\mathbf{v})) \;. \tag{9.6}$$

We have done all the work of proving equation (9.5) in Proposition 4.2, from which we see that

$$\begin{aligned} \mathrm{Hull}(\mathsf{C},\, FM(\mathsf{D}, \mathbf{v})) &= \mathrm{Hull}(\mathsf{C}, \mathsf{D}) \cap \mathbf{v}^\perp \\ &= \mathrm{Poly}(\mathsf{A}, \mathsf{B}) \cap \mathbf{v}^\perp \\ &= \mathrm{Poly}(\mathsf{A} + (\mathbf{v}), \mathsf{B}) \;. \end{aligned}$$

To prove equation (9.6), we most modify part of the reasoning of Proposition 4.2. First, the Hull is included in the Poly, because every generator of the former is contained in the latter. To prove inclusion the other way, let

$$\mathbf{x} \in \mathrm{Poly}(\mathsf{A}, \mathsf{B} + (\mathbf{v})) \ ;$$

that is,

$$\mathbf{x} \in \mathrm{Poly}(\mathsf{A}, \mathsf{B}) = \mathrm{Hull}(\mathsf{C}, \mathsf{D})$$

and also $(\mathbf{v}, \mathbf{x}) \leq 0$. Therefore

$$\mathbf{x} = \sum_i s_i \mathbf{c}_i + \sum_i t_i \mathbf{d}_i \ , \quad t_i \geq 0 \ ,$$

and also

$$\sum_i t_i (\mathbf{v}, \mathbf{d}_i) \leq 0 \ .$$

If n is the number of elements in D, let

$$\begin{aligned} P &= \{ i \,|\, 1 \leq i \leq n, \, (\mathbf{v}, \mathbf{b}_i) > 0 \} \, , \\ M &= \{ i \,|\, 1 \leq i \leq n, \, (\mathbf{v}, \mathbf{b}_i) < 0 \} \, , \\ Z &= \{ i \,|\, 1 \leq i \leq n, \, (\mathbf{v}, \mathbf{b}_i) = 0 \} \, . \end{aligned}$$

Also, let

$$\mathbf{x}_P = \sum_{i \in P} t_i d_i \ , \quad \mathbf{x}_M = \sum_{i \in M} t_i d_i \ ,$$

$$\mathbf{x}_Z = \sum_{i \in Z} t_i d_i \ , \quad \mathbf{x}_C = \sum_i s_i c_i \ .$$

If we define Λ_P and Λ_M by

$$\Lambda_P = \sum_{i \in P} t_i (\mathbf{v}, \mathbf{d}_i) = (\mathbf{v}, \mathbf{x}_P) \ ,$$

$$\Lambda_M = \sum_{i \in M} t_i (-(\mathbf{v}, \mathbf{d}_i)) = -(\mathbf{v}, \mathbf{x}_M) \ ,$$

then

$$\Lambda_M \geq \Lambda_P \geq 0 \ .$$

If $\Lambda_P = 0$, then $\mathbf{x}_P = 0$ and

$$\mathbf{x} = \mathbf{x}_C + \mathbf{x}_M + \mathbf{x}_Z \ ;$$

all the terms belong to $\mathrm{Hull}(\mathsf{C}, \, \mathsf{D}^- + FM(\mathsf{D}, \mathbf{v}))$.

If $\Lambda_P > 0$, then

$$\mathbf{x} = \mathbf{x}_C + \mathbf{x}_P + \mathbf{x}_Z + \mathbf{x}_M = \mathbf{x}' + \mathbf{x}''$$

where

$$\begin{aligned} \mathbf{x}' &= \mathbf{x}_C + \mathbf{x}_P + \mathbf{x}_Z + \frac{\Lambda_P}{\Lambda_M} \mathbf{x}_M \ , \\ \mathbf{x}'' &= \left(1 - \frac{\Lambda_P}{\Lambda_M} \right) \mathbf{x}_M \ . \end{aligned}$$

Now

$$\mathbf{x}' \in \mathrm{Hull}(\mathsf{C}, \mathsf{D}) \cap \mathbf{v}^{\perp} = \mathrm{Hull}(\mathsf{C}, FM(\mathsf{D}, \mathbf{v}))$$

by Proposition 4.2; and $\mathbf{x}'' \in \mathrm{Hull}(\mathsf{O}, \mathsf{D}^-)$. Therefore

$$\mathbf{x} \in \mathrm{Hull}(\mathsf{C},\, \mathsf{D}^- + FM(\mathsf{D}, \mathbf{v})) .$$

□

9.3. **Economized Fourier-Motzkin elimination.** At this point we seem to have convex polyhedral cones pretty well described. We understand their descriptions, both by generators and by constraints; and we now know, in principle, how to build up these descriptions from scratch. For instance, if we were given a set of generators of a cone and wanted to know its description by constraints, we could find it. We could start with the zero cone and add generators, one at a time, keeping track of the constraints as we went. Moreover, from §6 and Propositions 8.2 and 8.4, we know how to optimize the resulting double description. Our goal was to understand how to compute the descriptions of convex polyhedran abd perform operations on them, with reasonable efficiency. Have we not reached it?

No—almost, but not quite. The problem is that the process of Fourier-Motzkin elimination is combinatorially explosive. As we remarked at the end of §4, that process can turn n descriptors into anything up to $n^2/4$ descriptors. For any serious computation, we need to be able to economize here.

From Proposition 4.2, we know that the Fourier-Motzkin process, starting with the ray generators of a cone, arrives at a set of ray generators of the intersection of that cone with a hyperplane. We may optimize that set of generators, applying Propositions 8.2 and 8.4. This would get rid of redundant decriptors. It would be even better to avoid creating those redundant descriptors in the first place.

In this subsection, we prove two propositions which enable us to do just that. Let us suppose given a cone K with a double description

$$\mathcal{D} = ((\mathsf{A}, \mathsf{B}), (\mathsf{C}, \mathsf{D})) ,$$

so that

$$K = \mathrm{Hull}(\mathsf{A}, \mathsf{B}) = \mathrm{Poly}(\mathsf{C}, \mathsf{D}) .$$

Let us add a hyperplane constraint with descriptor $\mathbf{v}$, and suppose that $\mathbf{v}$ is orthogonal to all the line descriptors in A. Then, by Proposition 9.2, the resulting cone K' has this double description:

$$\mathcal{D}' = ((\mathsf{A}, FM(\mathsf{B}, \mathbf{v})), (\mathsf{C} + (\mathbf{v}), \mathsf{D})) ;$$

that is,

$$K' = \mathrm{Hull}(\mathsf{A}, FM(\mathsf{B}, \mathbf{v})) = \mathrm{Poly}(\mathsf{C} + (\mathbf{v}), \mathsf{D}) .$$

Recall that if $\mathsf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_m)$, then $FM(\mathsf{B}, \mathbf{v})$ contains those vectors $\mathbf{b}_h$ such that $(\mathbf{v}, \mathbf{b}_h) = 0$, and the vectors

$$\mathbf{b}'_{ij} = (\mathbf{v}, \mathbf{b}_i)\mathbf{b}_j - (\mathbf{v}, \mathbf{b}_j)\mathbf{b}_i \ , \quad (\mathbf{v}, \mathbf{b}_i) > 0 \ , \quad (\mathbf{v}, \mathbf{b}_j) < 0 \ .$$

We recall the notation $\sigma_i = \sigma_i(\mathcal{D})$ in Definition 8.1. Let us extend it by setting

$$\sigma_{ij} = \{k \in \{1, \ldots, n\} \,|\, (\mathbf{b}'_{ij}, \mathbf{d}_k) = 0\} \ .$$

Lemma 9.3. *Let K, $\mathcal{D}$, and $\mathbf{v}$ be as above. Let i and j be indexes of elements of B such that $(\mathbf{v}, \mathbf{b}_i) > 0$ and $(\mathbf{v}, \mathbf{b}_j) < 0$. Then $\sigma_{ij} = \sigma_i \cap \sigma_j$.*

Proof. First, it is easy to see that if $k \in \sigma_i \cap \sigma_j$, then $k \in \sigma_{ij}$. to prove the converse, suppose that $k \in \sigma_{ij}$. Then

$$\begin{aligned} 0 &= (\mathbf{v}, \mathbf{b}_i)(\mathbf{b}_j, \mathbf{d}_k) - (\mathbf{v}, \mathbf{b}_j)(\mathbf{b}_i, \mathbf{d}_k) \\ &= (\mathbf{v}, \mathbf{b}_i)(\mathbf{b}_j, \mathbf{d}_k) + (-(\mathbf{v}, \mathbf{b}_j)(\mathbf{b}_i, \mathbf{d}_k)) \ ; \end{aligned}$$

in the last expression, both summands are ≤ 0. Since their sum is 0, each summand must be 0, that is, $k \in \sigma_i \cap \sigma_j$. □

Before we apply Lemma 9.3 to economizing the description of K', we consider a special case. If $\mathbf{v}$ ls orthogonal to all the descriptors in A and B then it is redundant, and $K' = K$.

Proposition 9.4. *Let K, $\mathcal{D}$, $\mathbf{v}$, K' and $\mathcal{D}'$ be as above, and assume that $\mathbf{v}$ is orthogonal to all descriptors in A, but not to all descriptors in B. Let i and j be indexes of elements of B such that $(\mathbf{v}, \mathbf{b}_i) > 0$ and $(\mathbf{v}, \mathbf{b}_j) < 0$. If*

$$\#(\sigma_i \cap \sigma_j) < \dim \operatorname{aff}(K) - \dim \operatorname{lineal}(K) - 2 \ ,$$

then $\mathbf{b}'_{ij}$ is redundant for $\mathcal{D}'$.

Proof. By Proposition 8.2, $\mathbf{b}'_{ij}$ is redundant if

$$\#\sigma_{ij} < \dim \operatorname{aff}(K') - \dim \operatorname{lineal}(K') - 1 \ .$$

But, by Lemma 9.3, $\sigma_{ij} = \sigma_i \cap \sigma_j$; and the hypotheses on $\mathbf{v}$ imply that

$$\dim \operatorname{aff}(K') = \dim \operatorname{aff}(K) - 1 \ , \quad \dim \operatorname{lineal}(K') = \dim \operatorname{lineal}(K) \ .$$

□

Proposition 9.5. *Let K, $\mathcal{D}$, $\mathbf{v}$, K' and $\mathcal{D}'$ be as above, and assume that $\mathbf{v}$ is orthogonal to all descriptors in A, but not to all descriptors in B. Let i and j be indexes of elements of B such that $(\mathbf{v}, \mathbf{b}_i) > 0$ and $(\mathbf{v}, \mathbf{b}_j) < 0$. Then $\mathbf{b}'_{ij}$ is redundant for $\mathcal{D}'$ if and only if $\sigma_i \cap \sigma_j \subseteq \sigma_k$ for some k not equal to i or j.*

Proof. First, suppose that $\mathbf{b}'_{ij}$ is redundant. Then, by Proposition 8.4, there is an index k, or a pair of indexes (gh) distinct from (ij), such that $\sigma_{ij} \subseteq \sigma_k$ or $\sigma_{ij} \subseteq \sigma_{gh}$. But in the second case, $\sigma_{ij} \subseteq \sigma_g$ and $\sigma_{ij} \subseteq \sigma_h$.

Next, suppose that $\mathbf{b}'_{ij}$ is not redundant. We must show that, for every $k \in \{1, \ldots, m\}$ other than i and j, $\sigma_{ij} \nsubseteq \sigma_k$. Consider the following cases:

- $(\mathbf{v}, \mathbf{b}_k) = 0$.
  In this case, b_k is an element of $FM(\mathsf{B}, \mathbf{v})$, so by Proposition 8.4 applied to $\mathcal{D}'$, we have $\sigma_{ij} \nsubseteq \sigma_k$.
- $(\mathbf{v}, \mathbf{b}_k) > 0$, but $k \neq i$.
  In this case, by the same Proposition, $\sigma_{ij} \nsubseteq \sigma_{kj}$, that is,
  $$\sigma_i \cap \sigma_j \nsubseteq \sigma_k \cap \sigma_j \ .$$
  This relation easily implies $\sigma_i \cap \sigma_j \nsubseteq \sigma_k$.
- $(\mathbf{v}, \mathbf{b}_k) < 0$, but $k \neq j$.
  This case is like the previous one, applied to σ_{ik}.

□

Propositions 9.4 and 9.5 suggest an efficient way to construct an economical equivalent to $FM(\mathsf{B}, \mathbf{v})$: construct the sets σ_i, and examine the intersections σ_{ij}, to determine whether $\mathbf{b}'_{ij}$ will be redundant before going to the expense of constructing it at all. The careful reader may feel misgivings about this strategy, connected with the possibility that some of the rays constructed in $FM(\mathsf{B}, \mathbf{v})$ may be lineally equivalent. If, for instance, $\mathbf{b}'_{ij}$ and $\mathbf{b}'_{kl}$ are lineally equivalent, then testing σ_{ij} and σ_{kl} will reveal both to be redundant. Is it safe, then, to exclude both from the list of new descriptors?

Yes. Suppose that $\mathbf{b}'_{ij}$ and $\mathbf{b}'_{kl}$ are lineally equivalent, that is

$$\mathbf{b}'_{ij} = \lambda \mathbf{b}'_{kl} + \mathbf{x} \ , \quad \lambda > 0 \ , \quad \mathbf{x} \in \mathrm{lineal}(K) \ . \tag{9.7}$$

Then both $\mathbf{b}'_{ij}$ and $\mathbf{b}'_{kl}$ can be omitted, because they are lineally equivalent to a strictly positive combination of $\mathbf{b}'_{il}$ and $\mathbf{b}'_{kj}$. Indeed, we have

$$\frac{1}{v_i v_j}\mathbf{b}'_{ij} = \frac{1}{v_j}\mathbf{b}_j - \frac{1}{v_i}\mathbf{b}_i$$

and similar equations for $\mathbf{b}'_{kl}$, $\mathbf{b}'_{il}$, and $\mathbf{b}'_{kj}$. Therefore

$$\frac{1}{v_i v_j}\mathbf{b}'_{ij} + \frac{1}{v_k v_l}\mathbf{b}'_{kl} = \frac{1}{v_i v_l}\mathbf{b}'_{il} + \frac{1}{v_k v_j}\mathbf{b}'_{kj} \ ,$$

which together with equation (9.7) gives the desired lineal equivalence.

10. Implementation Notes

These notes accompany two Mathematica packages, “ConvexPolyhedra” and “ConvexCones.” These packages supply functions for constructing convex polyhedra in $\mathbb{R}^d$ and performing elementary operations on them. A small suite of test programs accompanies the packages. In this section, I describe, using Mathematica-related vocabulary, how these packages are organized.

The packages are contained in two Notebook files: ConvexPolyhedra.nb and ConvexCones.nb. It is assumed that these files are in a folder named Convex, which is contained in a folder on the $Path. You can modify this assumption, if you want, by changing the arguments of the BeginPackage commands in the two Notebooks.

“ConvexPolyhedra” defines the symbol ConvexPolyhedron, which is the Head of any expression that denotes a convex polyhedron. Several classes of functions are provided:

- to construct a polyhedron by giving lists of generators or lists of constraints;
- to modify a polyhedron by adding one or more generators or imposing one or more constraints;
- to retrieve the generators and constraints that describe a polyhedron;
- to perform other transformations on a polyhedron;
- to test for containment, subset relations, or equality of polyhedra.

The contents of ConvexPolyhedra can be made available by this command:

```
Needs["`Convex`ConvexPolyhedra`"];
```

In these notes, we showed how to relate convex polyhedra to convex cones, and after that worked mostly on convex cones. In the same way, the package ConvexPolyhedra does most of its work by calling on ConvexCones. The latter package has a similar structure to the former, with some simplifications.

The internal structure of an expression denoting a convex cone consists of its ambient dimension and its double description, in the form of four lists of descriptors: hyperplanes, halfspaces, lines, and rays. We take advantage of the symmetry between generators and constraints, so that adding a generator, and adding a constraint, are both imnplemented by calling an internal function, AddDescriptor. Functions for initializing a convex cone, or adding several generators or constraints, make repeaded calls to this function.

AddDescriptor is central to the implementation of ConvexCones. This function separates the problem of adding a new descriptor into three cases, just as we did in §9. For convenience, it first takes care of the special case that the new descriptor is redundant. Next in handles the "first case," as in §9.1; the details are taken care of by two calls to another internal function ShiftDescriptors. Finally, the "second case", as in §9.2, is handled; most of the details are in the economized Fourier-Motzkin process, which is taken care of by the internal function FME.

Optimization is done in two functions: FME and SimplifyCone. When one tries to optimize anything, there can be a trade-off between the quality of the result, and the amount of work needed to achieve this result. I decided that economizing every use of the Fourier-Motzkin process was best. It costs little, because some of the time spent deciding which new descriptors are redundant is made up by not computing the redundant descriptors. SimplifyCone, however, is not called within AddDescriptor; it is called once in every publicly visible function that needs it. Each application of AddDescriptor may make some existing descriptors redundant; but it does not cause a combinatorial explosion of redundant new descriptors.

References

1. A. Farkas, *Über die Theorie der einfachen Ungleichungen*, J. Reine Angew. Math. **124** (1902), 1–27.
2. Vilmos Komornik, *A simple proof of Farkas's lemma*, Amer. Math. Monthly **105** (1998), no. 10, 949–950.
3. Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, no. 152, Springer-Verlag, 1995.

19 Tillinghast Place, Buffalo, NY 14216
E-mail address: chenrich@monmouth.com