Orthogonal curvilinear coordinates

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December 17, 2004

1 Curvilinear coordinates

Let x_i with $i = 1, 2, 3$ be Cartesian coordinates of a point and let ξ_a with $a = 1, 2, 3$ be the corresponding curvilinear coordinates. We shall use ordinary Cartesian vector notation $\vec{x} = (x_1, x_2, x_3)$ for the Cartesian coordinates, but not for the curvilinear ones. The two sets of coordinates are connected by a bijective coordinate transformation

$$\vec{x} = \vec{f}(\xi_1, \xi_2, \xi_3) \tag{1}$$

The most important quantity is the infinitesimal vector element

$$d\vec{x} = \sum_a \vec{J}_a d\xi_a \tag{2}$$

where the $\vec{J}_a$ are the columns of the Jacobian

$$\vec{J}_a = \frac{\partial \vec{x}}{\partial \xi_a} \tag{3}$$

The Cartesian square norm of the infinitesimal element is

$$d\vec{x}^2 = \sum_{ab} g_{ab} d\xi_a d\xi_b \tag{4}$$

where

$$g_{ab} = \vec{J}_a \cdot \vec{J}_b \tag{5}$$

is the metric.

2 Orthogonality

A large subclass of interesting coordinate systems are orthogonal, which means that

$$g_{ab} = \vec{J}_a \cdot \vec{J}_b = 0 \qquad\qquad (a \neq b) \tag{6}$$

In that case it is better to write

$$\boxed{\frac{\partial \vec{x}}{\partial \xi_a} = h_a \vec{e}_a} \tag{7}$$

where h_a is a scale factor and $\vec{e}_a$ is a unit vector. These vectors form a local basis in each point

$$\vec{e}_a \cdot \vec{e}_b = \delta_{ab} \qquad\qquad \sum_a \vec{e}_a \vec{e}_a = \overleftrightarrow{1} \tag{8}$$

where the first equation expresses orthogonality of the basis and the second completeness. This permits normal vector and matrix algebra to be used in the curvilinear coordinates.

3 Basis derivatives

The derivatives $\partial\vec{e}_b/\partial\xi_a$ of the basis vectors play an important role. From $\vec{e}_b^2 = 1$ we get

$$\frac{\partial\vec{e}_b}{\partial\xi_a} \cdot e_b = 0 \tag{9}$$

for all a, b. This shows that all derivatives of a unit vector are orthogonal to the unit vector. Similarly from the symmetry of the second derivatives we get

$$\frac{\partial^2\vec{x}}{\partial\xi_a\partial\xi_b} = \frac{\partial(h_a\vec{e}_a)}{\partial\xi_b} = \frac{\partial(h_b\vec{e}_b)}{\partial\xi_a} \tag{10}$$

Expanding this becomes

$$\frac{\partial h_a}{\partial\xi_b}\vec{e}_a + h_a\frac{\partial\vec{e}_a}{\partial\xi_b} = \frac{\partial h_b}{\partial\xi_a}\vec{e}_b + h_b\frac{\partial\vec{e}_b}{\partial\xi_a} \qquad (a \neq b) \tag{11}$$

Expanding in the basis, and using that the derivative of a unit vector is orthogonal to the unit vector, this equation can only be fulfilled for

$$h_b\frac{\partial\vec{e}_b}{\partial\xi_a} = \frac{\partial h_a}{\partial\xi_b}\vec{e}_a + \lambda_{abc}\vec{e}_c \qquad (a \neq b \neq c) \tag{12}$$

where

$$\lambda_{abc} = \lambda_{bac} \qquad (a \neq b \neq c) \tag{13}$$

Dotting with $\vec{e}_c$ and using that $\vec{e}_b \cdot \vec{e}_c = 0$ we get

$$\lambda_{abc} = h_b\frac{\partial\vec{e}_b}{\partial\xi_a} \cdot \vec{e}_c = -h_b\frac{\partial\vec{e}_c}{\partial\xi_a} \cdot \vec{e}_b = -\frac{h_b}{h_c}\lambda_{acb} \tag{14}$$

Combining these two rules we get

$$\lambda_{abc} = -\frac{h_b}{h_c}\lambda_{acb} = -\frac{h_b}{h_c}\lambda_{cab} = \frac{h_a}{h_c}\lambda_{cba} = \frac{h_a}{h_c}\lambda_{bca} = -\lambda_{bac} = -\lambda_{abc} \tag{15}$$

Consequently we have $\lambda_{abc} = 0$ so that

$$\frac{\partial\vec{e}_b}{\partial\xi_a} = \frac{1}{h_b}\frac{\partial h_a}{\partial\xi_b}\vec{e}_a \qquad (a \neq b) \tag{16}$$

Dotting (11) with $\vec{e}_a$ we get

$$\frac{\partial\vec{e}_b}{\partial\xi_a} \cdot \vec{e}_a = \frac{1}{h_b}\frac{\partial h_a}{\partial\xi_b} \qquad (a \neq b) \tag{17}$$

and using $\vec{e}_a \cdot \vec{e}_b = 0$ this leads to

$$\frac{\partial\vec{e}_a}{\partial\xi_a} \cdot e_b = -\frac{1}{h_b}\frac{\partial h_a}{\partial\xi_b} \qquad (a \neq b) \tag{18}$$

Using completeness this becomes (as may be easily verified)

$$\boxed{\frac{\partial\vec{e}_b}{\partial\xi_a} = \frac{1}{h_b}\frac{\partial h_a}{\partial\xi_b}\vec{e}_a - \delta_{ab}\sum_c \frac{1}{h_c}\frac{\partial h_a}{\partial\xi_c}\vec{e}_c} \tag{19}$$

This concludes the analysis of derivatives of the basis vectors.

4 Vector operators

Derivative operators transform as

$$\frac{\partial}{\partial \xi_a} = \frac{\partial \vec{x}}{\partial \xi_a} \frac{\partial}{\partial \vec{x}} = h_a \vec{e}_a \cdot \vec{\nabla} \tag{20}$$

where $\nabla_i = \partial/\partial x_i$. Using completeness we get

$$\boxed{\vec{\nabla} = \sum_a \frac{\vec{e}_a}{h_a} \frac{\partial}{\partial \xi_a}} \tag{21}$$

or

$$\nabla_a = \frac{1}{h_a} \frac{\partial}{\partial \xi_a} \tag{22}$$

4.1 Gradient of scalar field

If f is a scalar field, then $\nabla_a f$ is the gradient in the local basis,

$$\boxed{(\vec{\nabla} f)_a = \nabla_a f} \tag{23}$$

4.2 Divergence of vector field

For the divergence of a vector field $\vec{v}$ with local components $v_a = \vec{e}_a \cdot \vec{v}$ we find

$$\begin{aligned}
\vec{\nabla} \cdot \vec{v} &= \sum_b \frac{\vec{e}_b}{h_b} \frac{\partial}{\partial \xi_b} \cdot \sum_a v_a \vec{e}_a = \sum_{ab} \frac{\vec{e}_b}{h_b} \cdot \frac{\partial (v_a \vec{e}_a)}{\partial \xi_b} = \sum_a \frac{1}{h_a} \frac{\partial v_a}{\partial \xi_a} + \sum_{ab} v_a \frac{\vec{e}_b}{h_b} \cdot \frac{\partial \vec{e}_a}{\partial \xi_b} \\
&= \sum_a \frac{1}{h_a} \frac{\partial v_a}{\partial \xi_a} + \sum_{ab} \frac{v_a}{h_a h_b} \frac{\partial h_b}{\partial \xi_a} - \sum_a \frac{v_a}{h_a^2} \frac{\partial h_a}{\partial \xi_a} \\
&= \sum_a \frac{1}{h_a} \frac{\partial v_a}{\partial \xi_a} + \sum_{a \neq b} \frac{v_a}{h_a h_b} \frac{\partial h_b}{\partial \xi_a}
\end{aligned}$$

Introducing $h = \prod_a h_a = h_1 h_2 h_3$ this may be written

$$\boxed{\vec{\nabla} \cdot \vec{v} = \frac{1}{h} \sum_a \frac{\partial (v_a h / h_a)}{\partial \xi_a}} \tag{24}$$

which is the most compact form of the divergence.

4.3 Curl of vector field

In the local system the curl becomes

$$\begin{aligned}
(\vec{\nabla} \times \vec{v})_a &= \sum_{bc} \vec{e}_a \cdot \frac{\vec{e}_b}{h_b} \frac{\partial}{\partial \xi_b} \times v_c \vec{e}_c \\
&= \sum_{bc} \epsilon_{abc} \frac{1}{h_b} \frac{\partial v_c}{\partial \xi_b} + \sum_{bc} \frac{v_c}{h_b} \vec{e}_a \cdot \vec{e}_b \times \frac{\partial \vec{e}_c}{\partial \xi_b} \\
&= \sum_{bc} \epsilon_{abc} \frac{1}{h_b} \frac{\partial v_c}{\partial \xi_b} + \sum_{bc} \frac{v_c}{h_b} \vec{e}_a \cdot \vec{e}_b \times \left(\frac{1}{h_c} \frac{\partial h_b}{\partial \xi_c} \vec{e}_b - \delta_{bc} \sum_d \frac{1}{h_d} \frac{\partial h_b}{\partial \xi_d} \vec{e}_d \right) \\
&= \sum_{bc} \epsilon_{abc} \left(\frac{1}{h_b} \frac{\partial v_c}{\partial \xi_b} - \frac{v_b}{h_b h_c} \frac{\partial h_b}{\partial \xi_c} \right) = \sum_{bc} \epsilon_{abc} \left(\frac{1}{h_b} \frac{\partial v_c}{\partial \xi_b} + \frac{v_c}{h_b h_c} \frac{\partial h_c}{\partial \xi_b} \right)
\end{aligned}$$

This can be combined into

$$\boxed{(\vec{\nabla} \times \vec{v})_a = \sum_{bc} \frac{1}{h_b h_c} \frac{\partial (v_c h_c)}{\partial \xi_b}} \tag{25}$$

4.4 Gradient of vector field

The vector field gradient $\vec{\nabla}\vec{v}$ is a tensor with the following components in the local basis

$$\begin{aligned}
(\vec{\nabla}\vec{v})_{ab} &= \frac{1}{h_a} \frac{\partial \vec{v}}{\partial \xi_a} \cdot \vec{e}_b = \frac{1}{h_a} \sum_c \frac{\partial (v_c \vec{e}_c)}{\partial \xi_a} \cdot \vec{e}_b \\
&= \frac{1}{h_a} \frac{\partial v_b}{\partial \xi_a} + \frac{1}{h_a} \sum_c v_c \frac{\partial \vec{e}_c}{\partial \xi_a} \cdot \vec{e}_b \\
&= \frac{1}{h_a} \frac{\partial v_b}{\partial \xi_a} + \frac{1}{h_a} \sum_c v_c \left(\frac{1}{h_c} \frac{\partial h_a}{\partial \xi_c} \vec{e}_a - \delta_{ac} \sum_d \frac{1}{h_d} \frac{\partial h_a}{\partial \xi_d} \vec{e}_d \right) \cdot \vec{e}_b
\end{aligned}$$

$$\boxed{(\vec{\nabla}\vec{v})_{ab} = \frac{1}{h_a} \frac{\partial v_b}{\partial \xi_a} + \frac{\delta_{ab}}{h_a} \sum_c \frac{v_c}{h_c} \frac{\partial h_a}{\partial \xi_c} - \frac{v_a}{h_a h_b} \frac{\partial h_a}{\partial \xi_b}} \tag{26}$$

One may immediately verify that its trace equals the divergence.

Specializing to diagonal and non-diagonal elements we get

$$(\vec{\nabla}\vec{v})_{ab} = \frac{1}{h_a} \frac{\partial v_b}{\partial \xi_a} - \frac{v_a}{h_a h_b} \frac{\partial h_a}{\partial \xi_b} \qquad (a \neq b) \tag{27}$$

$$(\vec{\nabla}\vec{v})_{aa} = \frac{1}{h_a} \frac{\partial v_a}{\partial \xi_a} + \sum_{c \neq a} \frac{v_c}{h_a h_c} \frac{\partial h_a}{\partial \xi_c} \qquad (a = b) \tag{28}$$

4.5 Divergence of tensor

It is often necessary to calculate the divergence of a tensor $\vec{\nabla} \cdot \overleftrightarrow{t}$ in curvilinear coordinates. We find

$$(\vec{\nabla} \cdot \overleftrightarrow{t})_a = \sum_{bcd} \frac{\vec{e}_b}{h_b} \cdot \frac{\partial (t_{cd} \vec{e}_c \vec{e}_d)}{\partial \xi_b} \cdot \vec{e}_a \tag{29}$$

Expanding the sum we get

$$\begin{aligned}
(\vec{\nabla} \cdot \overleftrightarrow{t})_a &= \sum_{bc} \frac{\vec{e}_b}{h_b} \cdot \frac{\partial (t_{ca} \vec{e}_c)}{\partial \xi_b} + \sum_{bd} \frac{t_{bd}}{h_b} \frac{\partial \vec{e}_d}{\partial \xi_b} \cdot \vec{e}_a \\
&= \vec{\nabla} \cdot \vec{t}_a + \sum_b \frac{1}{h_a h_b} \left(t_{ab} \frac{\partial h_a}{\partial \xi_b} - t_{bb} \frac{\partial h_b}{\partial \xi_a} \right)
\end{aligned}$$

Using the divergence of a vector this becomes

$$\boxed{(\vec{\nabla} \cdot \overleftrightarrow{t})_a = \sum_b \frac{1}{h_b} \frac{\partial t_{ba}}{\partial \xi_b} + \sum_{b \neq c} \frac{t_{ba}}{h_b h_c} \frac{\partial h_c}{\partial \xi_b} + \sum_{b \neq a} \frac{t_{ab}}{h_a h_b} \frac{\partial h_a}{\partial \xi_b} - \sum_{b \neq a} \frac{t_{bb}}{h_a h_b} \frac{\partial h_b}{\partial \xi_a}} \tag{30}$$