Euclidean Curve Theory

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Mathenatica v.9.0.1.0 or v. 11.1.1.0,, and v.12.0.

▪ Summary

In this notebook we develop *Mathematica* tools for the Euclidean differential geometry of curves. We construct Modules for the calculation of all Euclidean invariants like arc length, curvatures, and Frenet formulas in the plane, the 3-space, and in n-dimensional Euclidean spaces. As an application we show that the curves of constant curvatures in the 4-dimensional Euclidean space are isogonal trajectories of certain circular tori and visualize them by stereographic projection. A short presentation of Euclidean curve theory as it is used in the present notebook is given in my paper [ECG] which is included in the zipfile eudiffgeov4.zip, or may be downloaded from my my homepage, In the book [G06], see also [G94], Alfred Gray presented Euclidean differential geometry with many applications of *Mathematica*. I am very much obliged to Alfred Gray who already in 1988 introduced me to Stephen Wolfram's program Mathematica. Many thanks also to Michael Trott for valuable hints improving the effectivity of the symbolic calculations contained in this notebook.

Revising this notebook I added subsection 4.5 about osculating circles and osculating spheres of a curve in the Euclidean space. The basic Mathematica package has been corrected and enlarged. I tested the notebook with *Mathematica* v. 9.0.1, v. 10, v.11.1.1, and v. 12.0.

▪ Keywords

curve, smooth, regular, singular, motion, velocity, arc length, tangent, binormal, principal normal, Frenet formulas, curvatures, torsion, graph,

osculating circle, osculating sphere, helix, spiral, 1-parameter motion group, orbits, torus, isogonal trajectory.

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Initialization

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1. List of Symbols and their Usages

In this Section one finds tables of all the symbols contained in the loaded packages and those introduced in the Global Context.. To get the usages click on the name! If this does not work, enable Dynamic Updating in the Evaluation Menu.

- **1.1. Symbols in the Package euvecv2.m**
- **1.2. Symbols in the Package tensalgv3.m**
- **1.3. Symbols in the Package eudiffgeov4.m**
- **1.4. Symbols in the Package Curves.m**
- **1.5. Symbols in the Global Context**

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2. Regular Curves. Examples in the Euclidean Plane

In this Section we develop basic concepts of the differential geometry of curves; as example we consider curves in the Euclidean plane.

- **2.1. Definitions**

- **2.1.1. Regular Curves. Tangents**
- **2.1.2. Arc Length**

- **2.2. Curvature. Graphs. Spirals**

- **2.3. Frenet Formulas for Plane Curves**

- **2.3.1. The Fundamental Theorem**
- **2.3.2. Examples**
- **2.3.3. Curves Represented with an Arbitrary Parameter**

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3. Curves in the Euclidean Space

Now we consider curves in the three-dimensional Euclidean space. Our aim is to describe the basic invariants of the curves, the curvature and the torsion, and create Mathematica Modules to calculate them.

- **3.1. Settings. The General Curve: curve3D**

- **3.2. Frenet Formulas for Space Curves**

- **3.3. Applications**

- **3.3.1. Plane Curves as Special Space Curves**
- **3.3.2. Helices and 1-Parameter Subgroups of the Euclidean Group**
- **3.3.3. Very Flat Curves**

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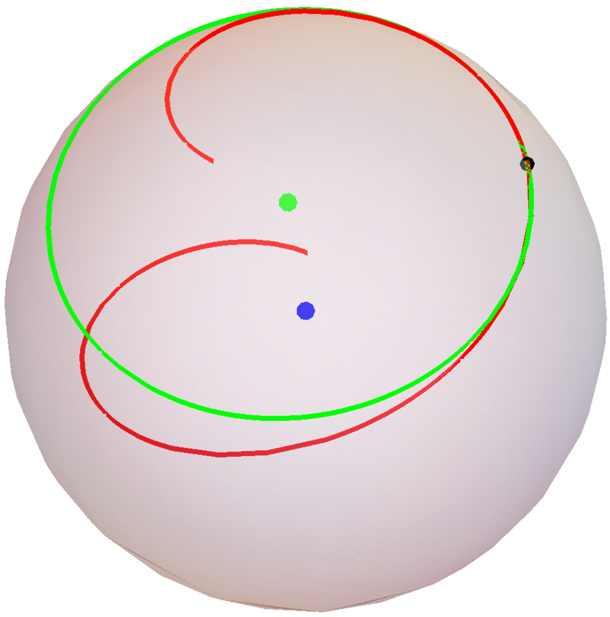
4. Osculating Circle and Osculating Sphere

Using the Frenet frame of a curve in the n-dimensional Euclidean space we construct Modules to calculate the osculating circle and the osculating sphere of the curve.



This picture shows a piece of a curve (red), one of its points (black), the osculating circle with its center (green), and the osculating sphere with its center (blue) at this point, see subsection 4.2.

- **4.1. The Osculating Circle**
- **4.2. The Osculating Sphere**

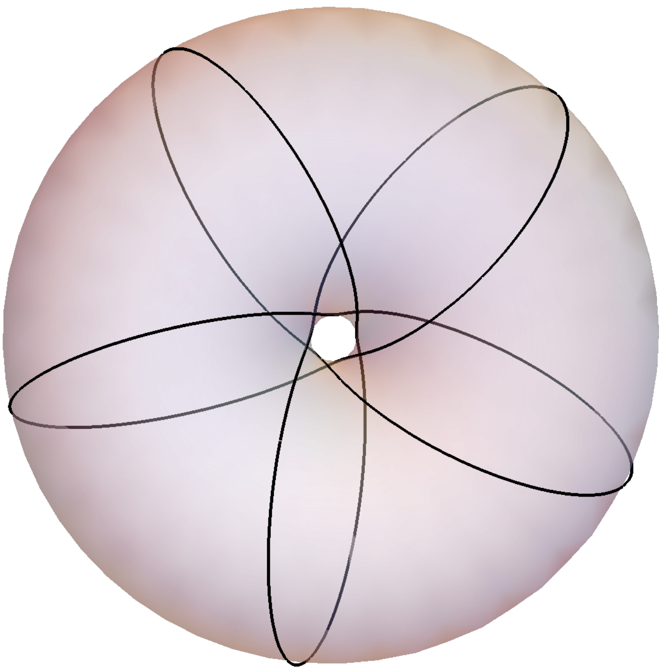
- **4.3. Examples**

In this Subsection we consider three examples. Use the definitions of space curves inA. Gray's package Curves3D.m, see also Subsection6.1.

- **4.3.1. genhelix**
- **4.3.2. ast3d**
- **4.3.3. sinn**

5. Curves of Constant Curvatures and 1-Parameter Motion Groups

Applying the built-in *Mathematica* function MatrixExp the curves of constant curvatures are treated here as orbits of 1-parameter motion groups. In particular it is shown that the orbits of maximal rank in the four-dimensional space are the isogonal trajectories of the family of generating circles of tori.



The picture shows the stereographic projection of a torus with a curve of constant curvature in the 4-dimensional Euclidean space.

- **5.1. Screw Motions in the Euclidean 3-Space**
- **5.2. Curves of Constant Curvatures in the Euclidean 4-Space**
- **5.3. The Shape of the Orbits of Rank 4**
- **5.4. Isogonality**

■ 5.5. Higher Dimensions

6. Examples. DSolve. NDSolve

■ 6.1. Alfred Gray's Space Curves

In this subsection we plot some curves and calculate their invariants. The user may continue considering other curves of Gray's list or creating new curves.

- **6.1.1. Initialization**
- **6.1.2. Astroid in 3D**
- **6.1.3. Elliptical Helix**
- **6.1.4. The Twicubic**
- **6.1.5. Viviani Curves**
- **6.1.6. Power Functions**

■ 6.2. Solution of the Frenet Equations

In this experimental Section we try to use the *Mathematica* built-in programs Dsolve and NDSolve to obtain solutions of the Frenet equations with given curvature function.

- **6.2.1. DSolve**
- **6.2.2. NDSolve**
- **6.2.3. Further Examples Using NDSolve**

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