Green's Functions Computation

User's manual

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1. Language

Code written in Mathematica 8.0.1.0

Version: november 2011.

2. Environment

Notebook Mathematica.

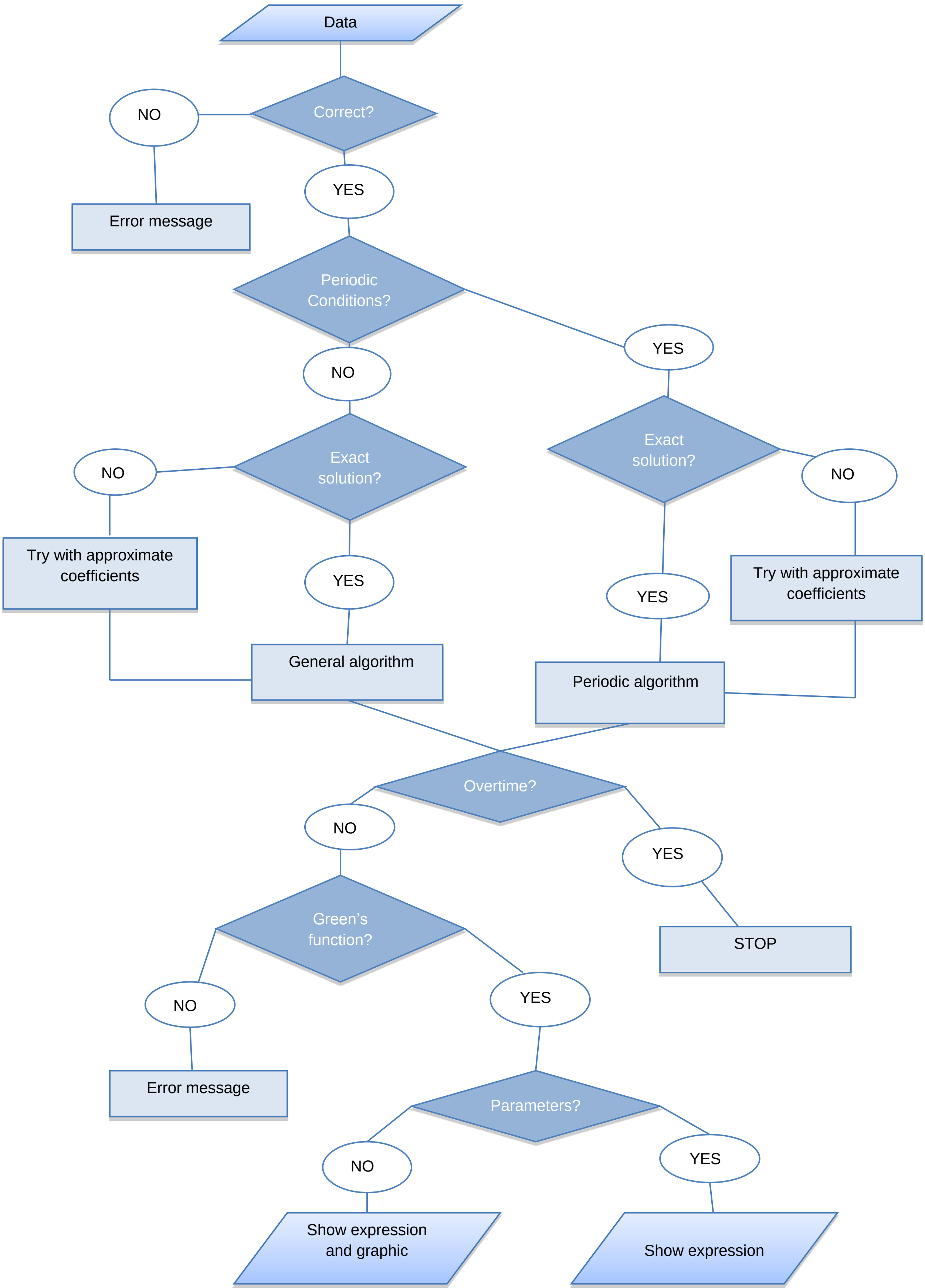
3. Name of the file

GFC.nb

4. Abstract

This Mathematica package provides a tool valid for calculating the explicit expression of the Green's function related to a n^{th} – order linear ordinary differential equation, with constant coefficients, coupled with two – point linear boundary conditions.

5. Flowchart



Data
Correct?
NO
Error message
YES
Periodic Conditions?
NO
YES
Exact solution?
NO
Try with approximate coefficients
YES
General algorithm
Exact solution?
NO
Try with approximate coefficients
YES
Periodic algorithm
Overtime?
NO
YES
STOP
Green's function?
NO
Error message
YES
Parameters?
NO
Show expression and graphic
YES
Show expression

6. User's manual

The program *Green's Functions Computation* calculates the Green's function, $G(t,s)$, from the boundary value problem given by a linear n^{th} - order ODE with constant coefficients

$$u^{(n)}(t) + c_1\, u^{(n-1)}(t) + c_2\, u^{(n-2)}(t) + \ldots + c_n\, u(t) = \sigma(t), \qquad t \in [a,b], \quad (1)$$

together with the boundary conditions

$$\sum_{j=0}^{n-1} \alpha_i^j u^{(j)}(a) + \beta_i^j u^{(j)}(b) = 0, \qquad 1 \le i \le n. \quad (2)$$

Now, we present the definition and the main property of the Green's function. For a more detailed study, the reader is referred to the bibliography.

Definition. A Green's function for problem (1)-(2) is any function $G(t,s)$ that satisfies the following axioms:

(G1) $G(t,s)$ is defined on the square $[a,b] \times [a,b]$.

(G2) For $k = 0,1,\ldots,n-2$, the partial derivatives $\frac{\partial^k G}{\partial t^k}$ exist and they are continuous on $[a,b] \times [a,b]$.

(G3) $\frac{\partial^{n-1} G}{\partial t^{n-1}}$ and $\frac{\partial^n G}{\partial t^n}$ exist and are continuous on the triangles $a \le s < t \le b$ and $a \le t < s \le b$.

(G4) For each $t \in (a,b)$ there exist the lateral limits $\frac{\partial^{n-1} G(t,t^+)}{\partial t^{n-1}}$ and $\frac{\partial^{n-1} G(t,t^-)}{\partial t^{n-1}}$, and moreover

$$\frac{\partial^{n-1} G(t,t^+)}{\partial t^{n-1}} - \frac{\partial^{n-1} G(t,t^-)}{\partial t^{n-1}} = -1.$$

(G5) For each $s \in (a,b)$, the function $G(\cdot,s)$ is a solution of the differential equation (1), with $\sigma \equiv 0$, on both intervals $[a,s)$ and $(s,b]$.

(G6) For each $s \in (a,b)$, the function $G(\cdot,s)$ satisfies the boundary conditions (2).

The next result shows the importance of the Green's function in solving boundary value problems.

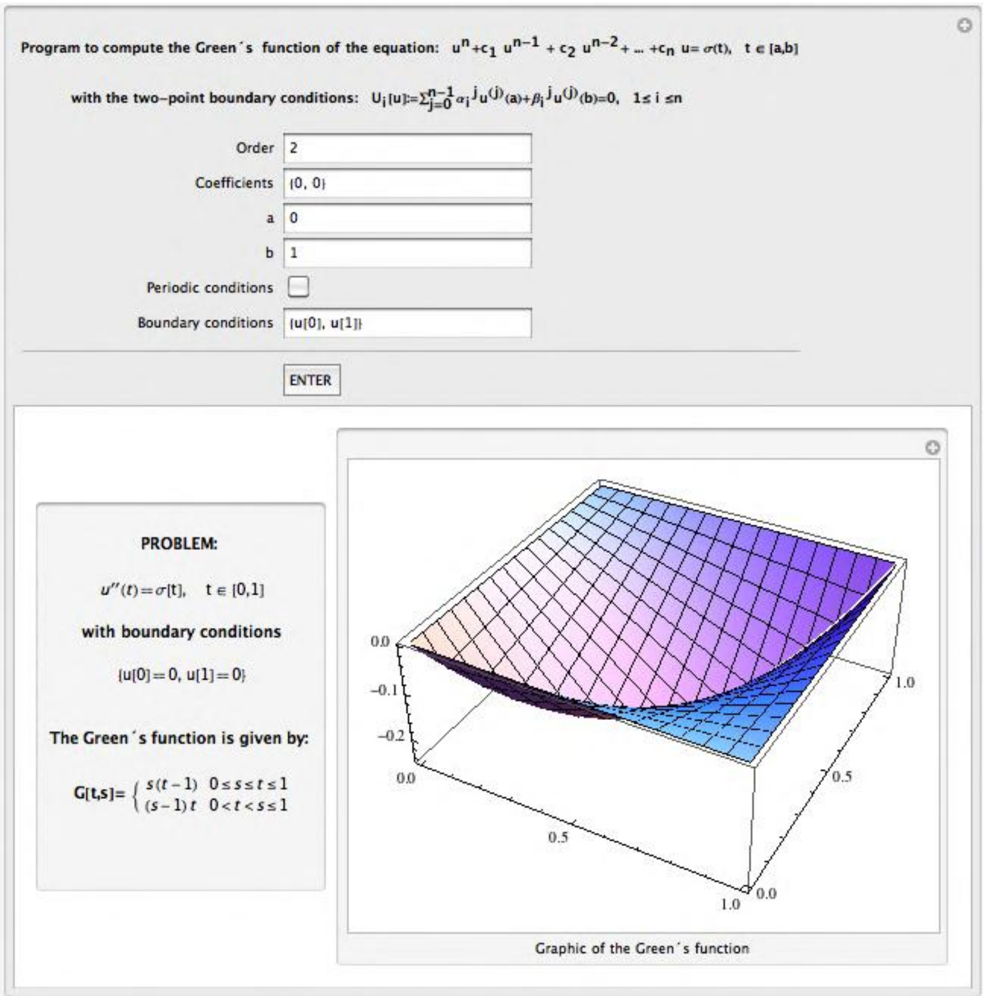
Theorem. Let us suppose that problem (1)-(2) with $\sigma \equiv 0$ has only the trivial solution. Then there exists a unique Green's function $G(t,s)$ and for each continuous function σ the unique solution of problem (1)-(2) is given by the expression

$$u(t) = \int_a^b G(t,s)\sigma(s)ds.$$

6.1. The *Mathematica* notebook

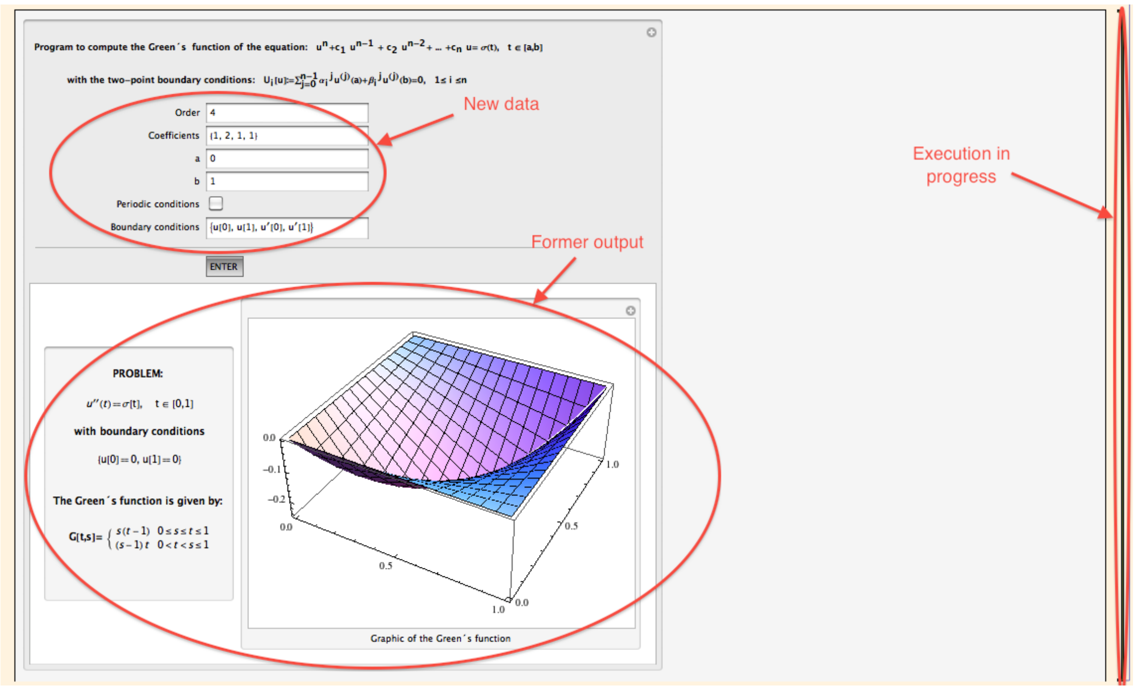
The *Green's Functions Computation* is a *Mathematica* notebook with a dynamic environment. In order to run the program *Wolfram Mathematica* is needed on the user's computer. This notebook is intended for version 8.0.1.0 but it also works on less recent versions.

To opening the notebook, the "Enabled dynamic" button must be pushed. After that it will appear the following execution corresponding to a simple second order problem.



The file is protected against writing, so if the user wants to save the file, he/she should use the option “Save As” from the menu “File”.
The program has two different areas: *input* and *output,* both separated by the “Enter” button. The data must be entered in the input area by using the *Mathematica* notation. After that, it must be pushed “Enter” to start the execution and the result will be shown on the *output* area, which is divided again in two areas: *analytical output* and *graphical output*. Notice that while the program is running the “Enter” button will appear pressed.

The program is inside a *Mathematica* cell, which is framed by a vertical line on the right of the screen. When that line is bolded, the program is running. Until a new execution is completed, the output will be the corresponding to the previous one.

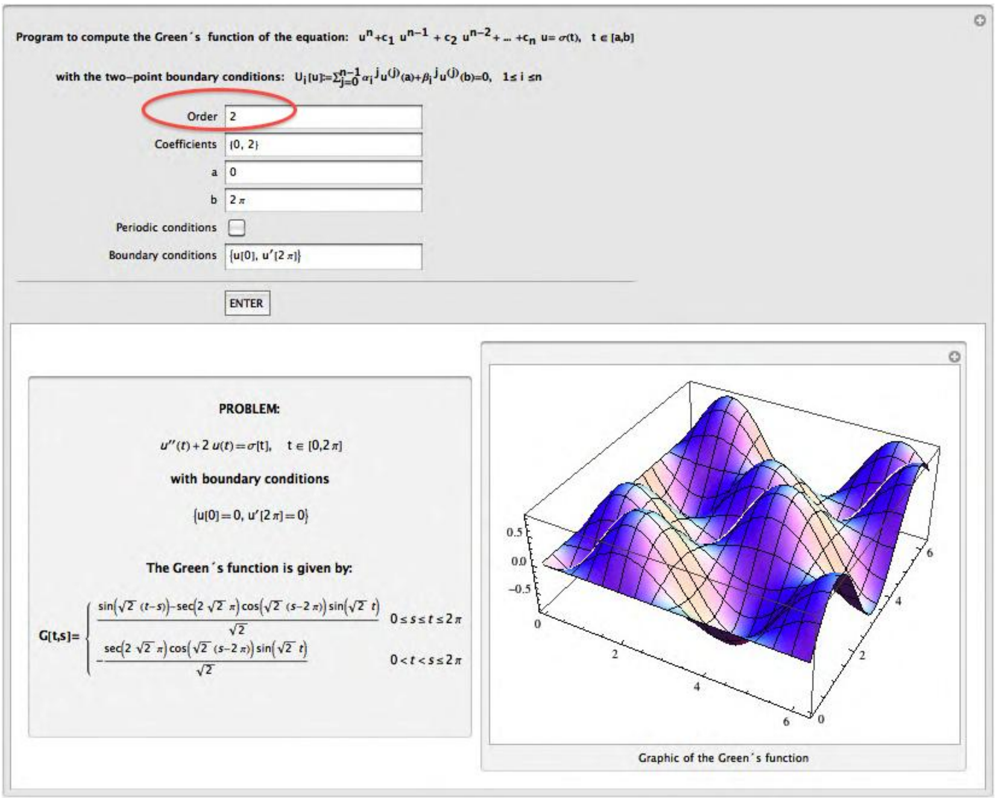


6.2. The data

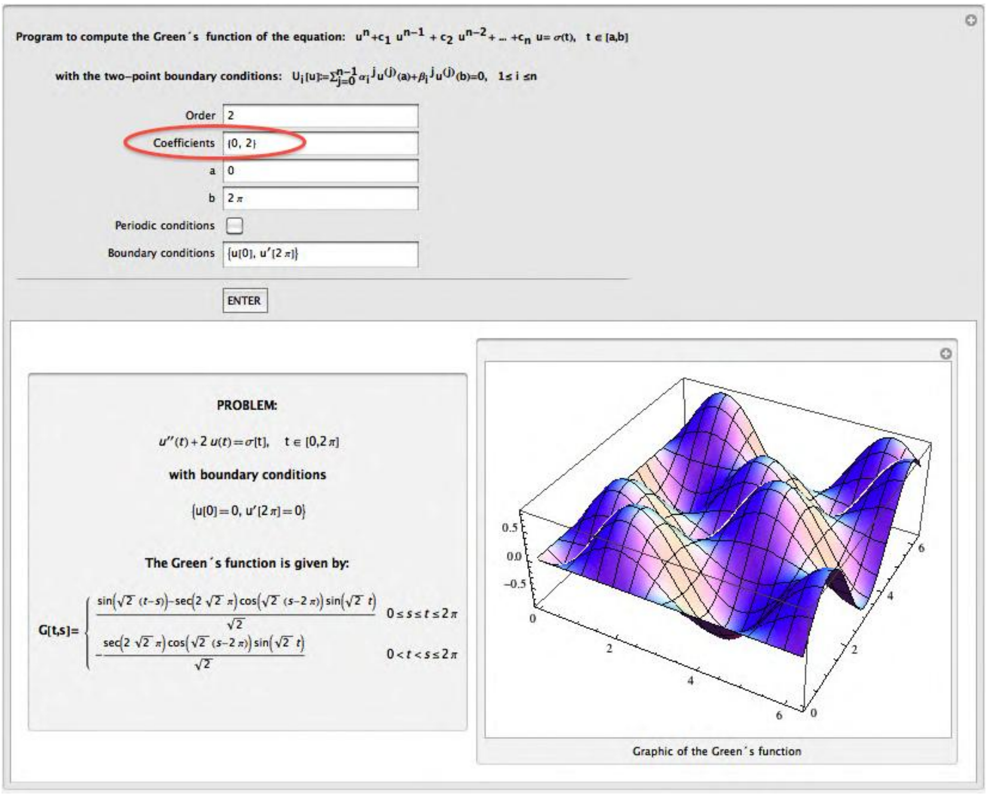
To start an execution the following boxes must be completed: **Order**, **Coefficients**, **a**, **b**, **Periodic Conditions** and **Boundary conditions.** Notice that all the data have to be entered in *Mathematica* syntax. Some examples are showing in the following table:

	Mathematica syntax	Examples
Power	^	m^2
Multiply	* or *empty space*	c*x, c x
Divide	/	1/2, -3/7
Constants	Pi, E (for other constants see the *Mathematica* help)	2 Pi, E^(-1)
Functions	Sin[x], Cos[x], Tan[x], Log[x], Sqrt[x] (for other functions see the *Mathematica* help)	2 Sin[3 x], Sqrt[2]
Lists, vectors	{ , , … }	{1,3,2}, {u[0],u'[Pi]}
Derivative	'	u'[0], u''[2], u'''[2*Pi]
Grouping terms	( )	m^(2*x)

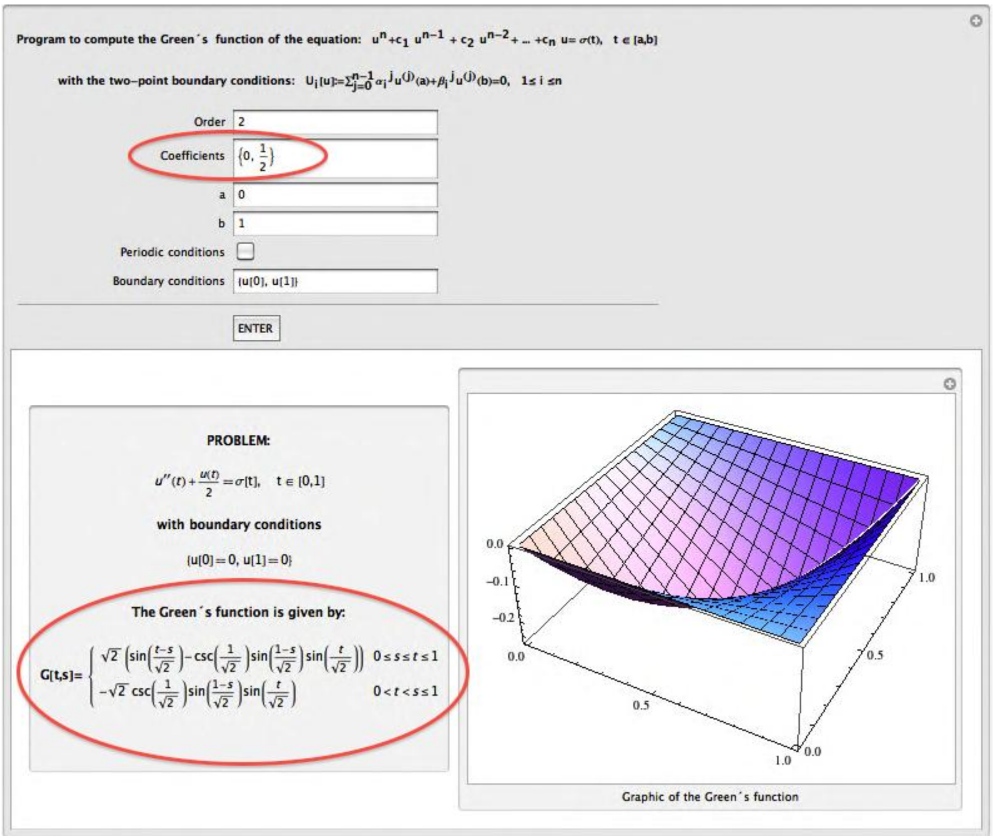
The first box, **Order**, is referred to the order of the differential equation, so it must be a natural number. Moreover the order will set both the number of coefficients as well as the number of boundary conditions.



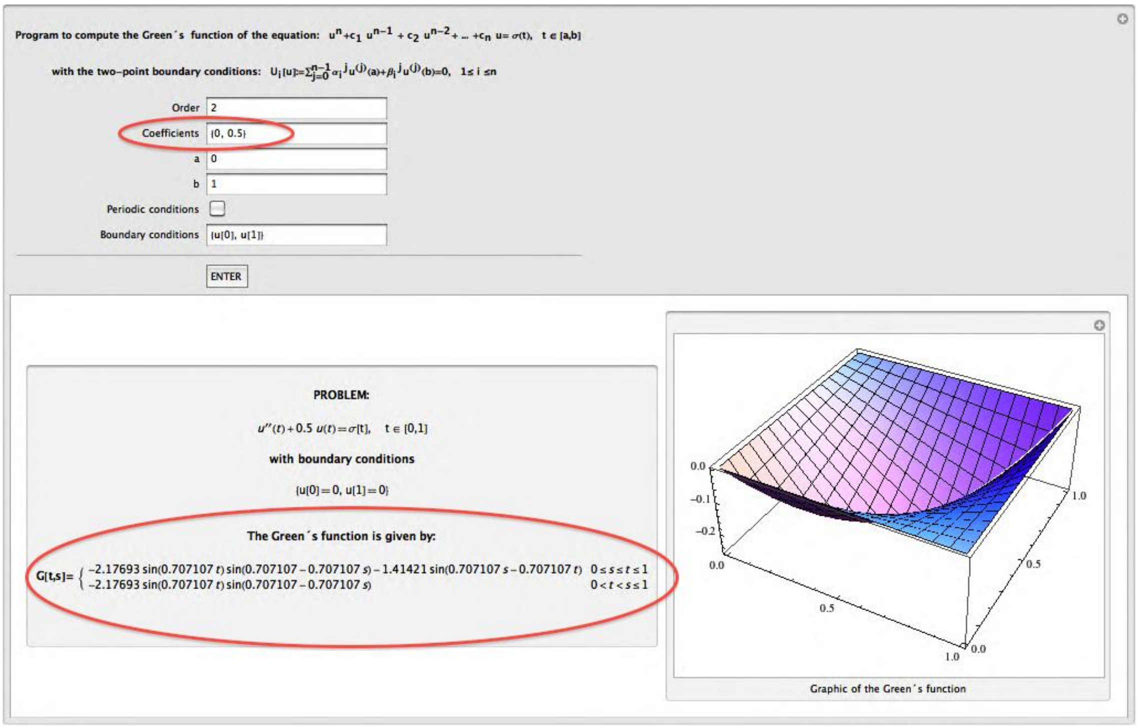
The second step consists on introducing the coefficients vector of the differential equation, $\{c_1,\ldots,c_n\}$, with the same length as the order of the equation.



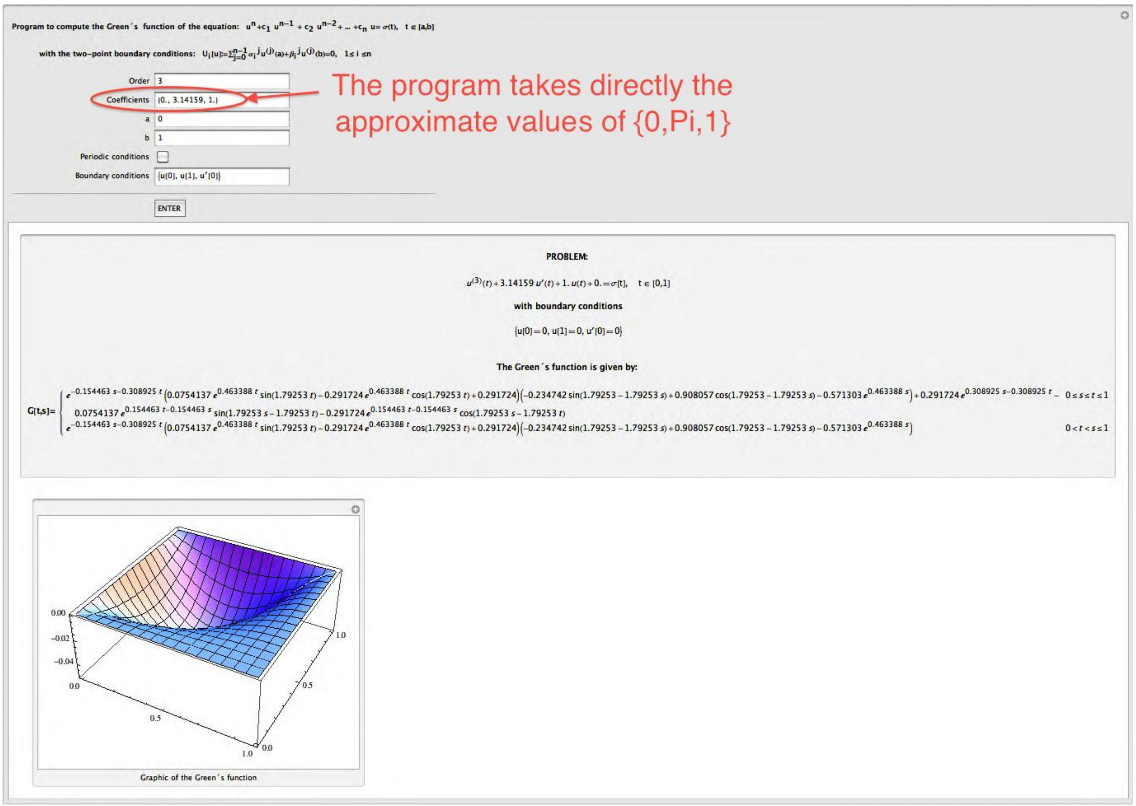
The coefficients of the differential equation must be real constants. If those coefficients are exact (for instance: integer numbers (-1,0,5,...), fractions with integer numbers (1/3,-2/45,...) or exact real numbers (Pi, Sqrt[2],...) the computations made by *Mathematica* will be exact as the final result.



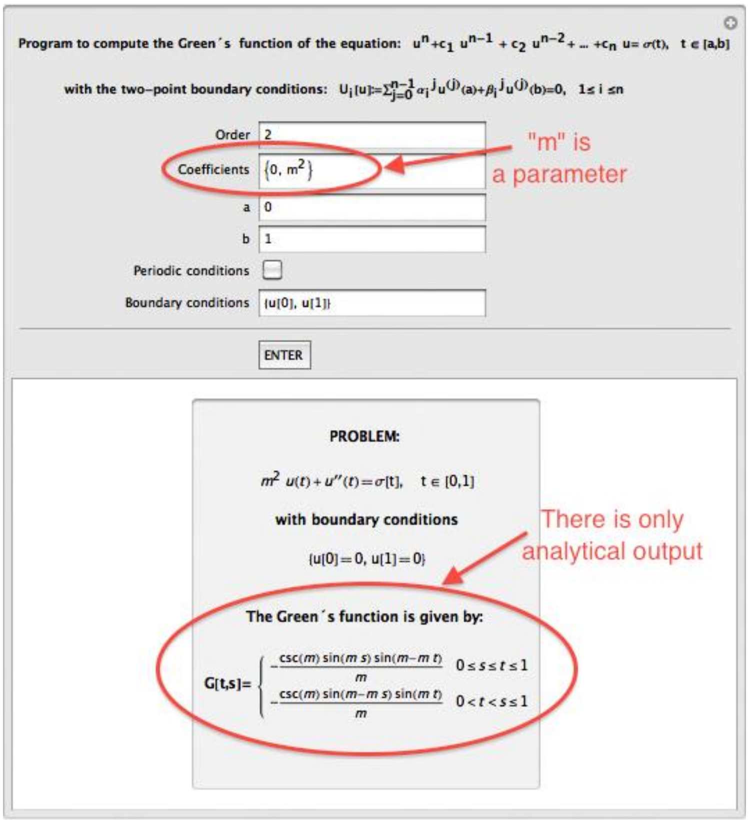
However if any of the introduced constants is an approximate number (for instance: 1.0, 3.14159, Sqrt[2.0], -0.5,...), the final result will be also approximate.



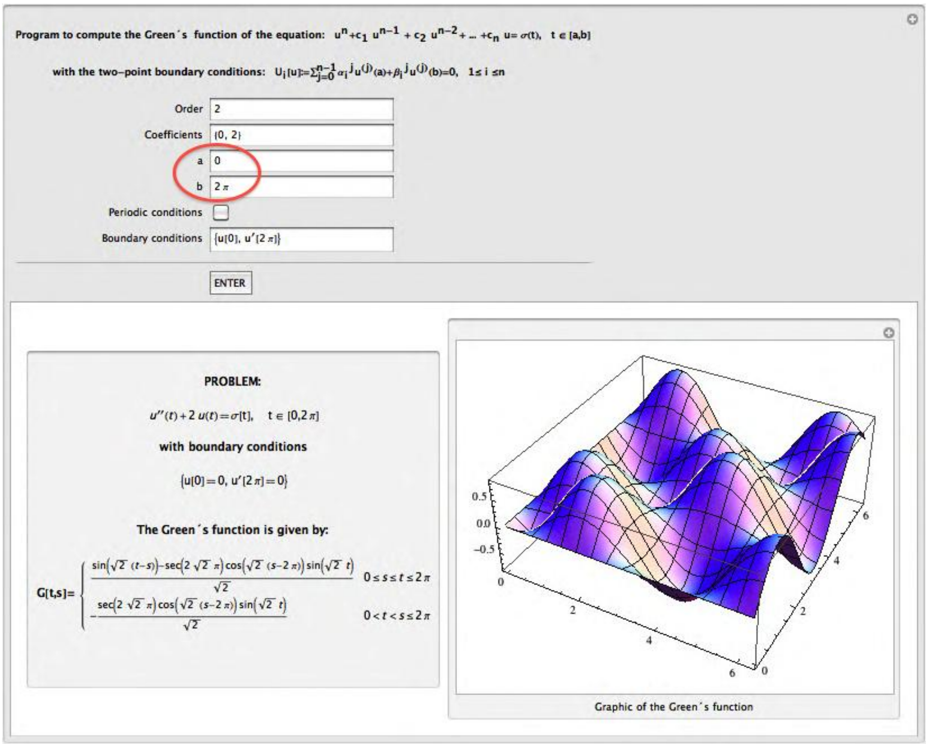
Sometimes, when using exact numbers, *Mathematica* is not able to solve the corresponding problem. The program detects this fact and, in these situations, it considers the approximate values of the introduced coefficients. For instance, if we want to obtain the Green's function of the problem u'''+ π u'+u= σ, t on [0,1], u(0)=u(1)=u'(0)=0, even entering the coefficients vector {0, π,1}, the obtained solution will be the following:



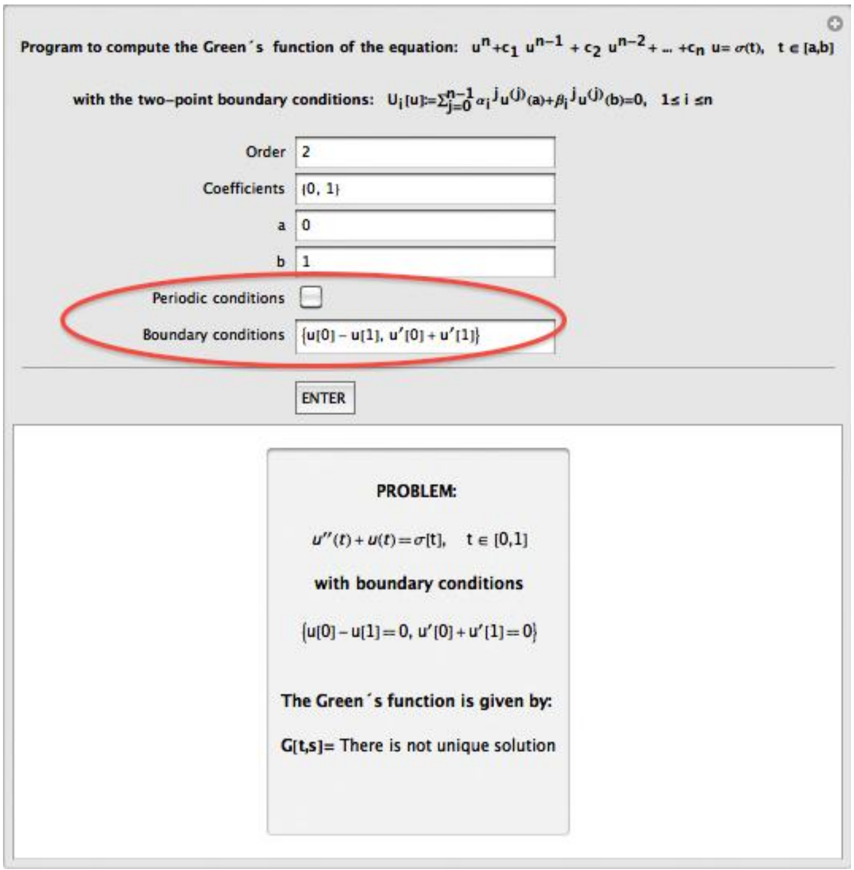
Entering parameters is allowed in the **Coefficients** box, but in this case the output will be only analytical and not graphical.



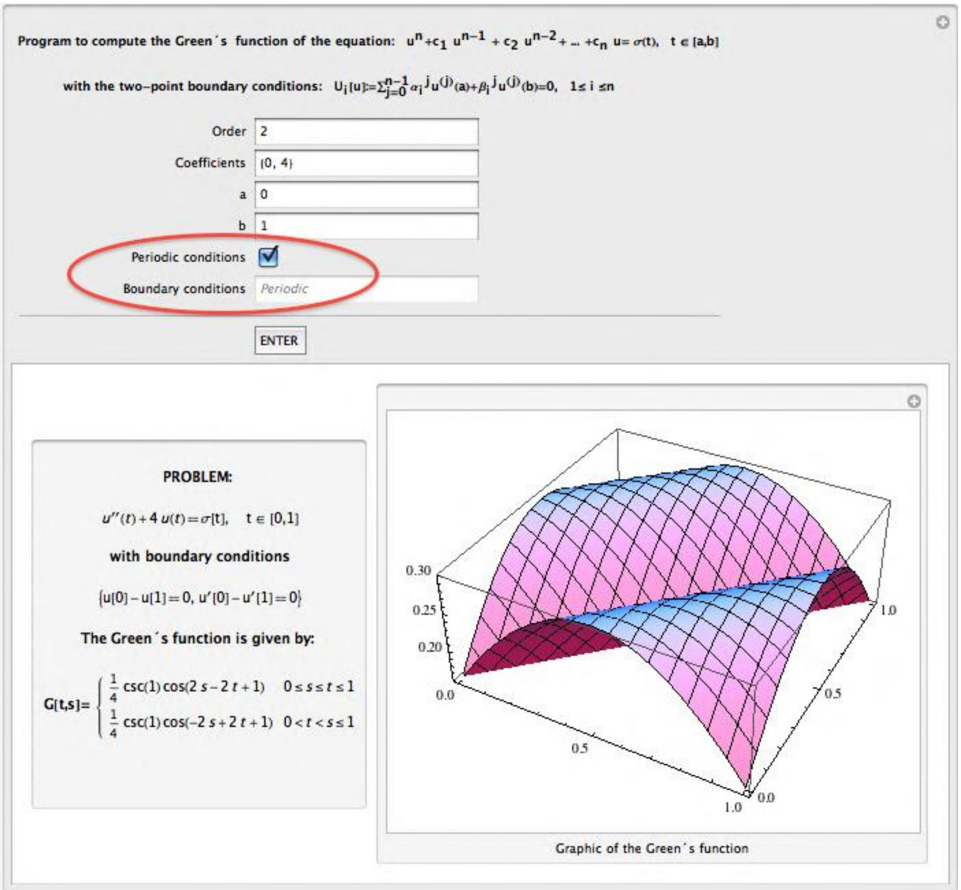
In the third and fourth boxes the user must enter the endpoints of the interval. These values could be also parameters (but again in this case the output would be only analytical).



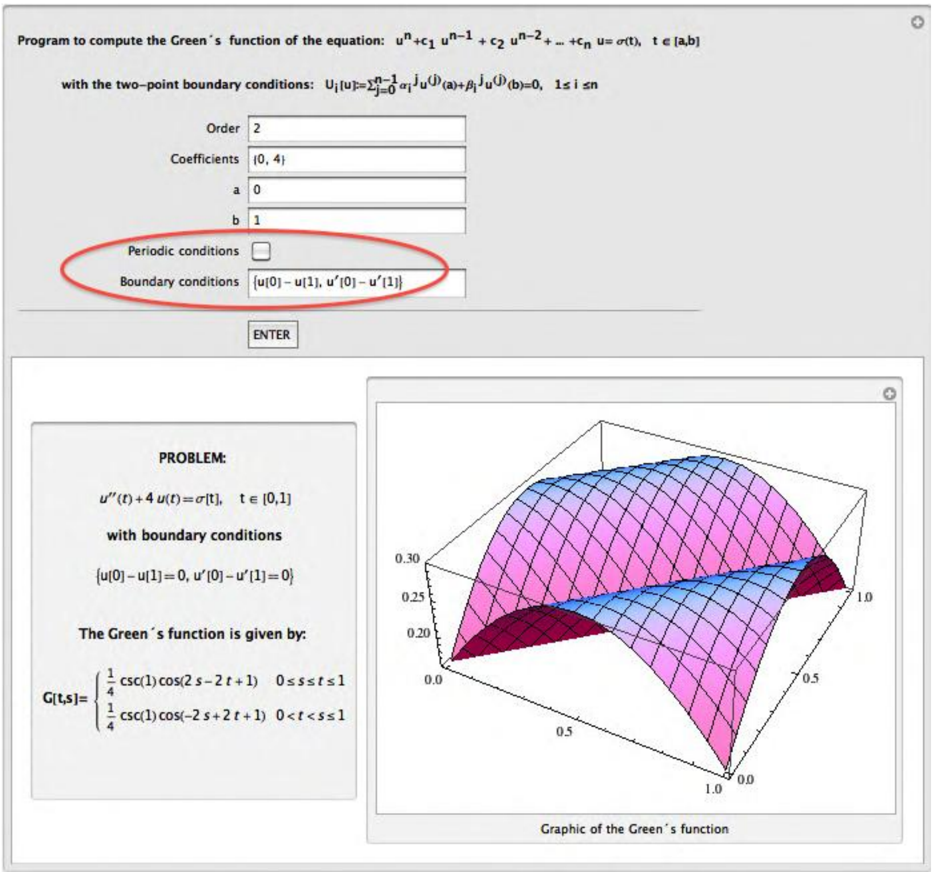
Last box is for entering the boundary conditions. They must be a vector of the same length as the order, *n*, of the differential equation. Moreover the boundary conditions must depend linearly on *u* and its derivatives up to the *n-1* order and they must be evaluated at the endpoints of the interval. The vector with the boundary conditions will be matched to zero by the program. For instance, to use the boundary conditions *u(0)=u(1), u'(0)= – u'(1),* the vector {u[0] – u[1],u'[0]+u'[1]} must be entered.



The option **Periodic conditions** is unmarked by default. By marking it, not only the considered boundary conditions are the periodic ones, but also a specific algorithm is used by the program to find out the Green's function.

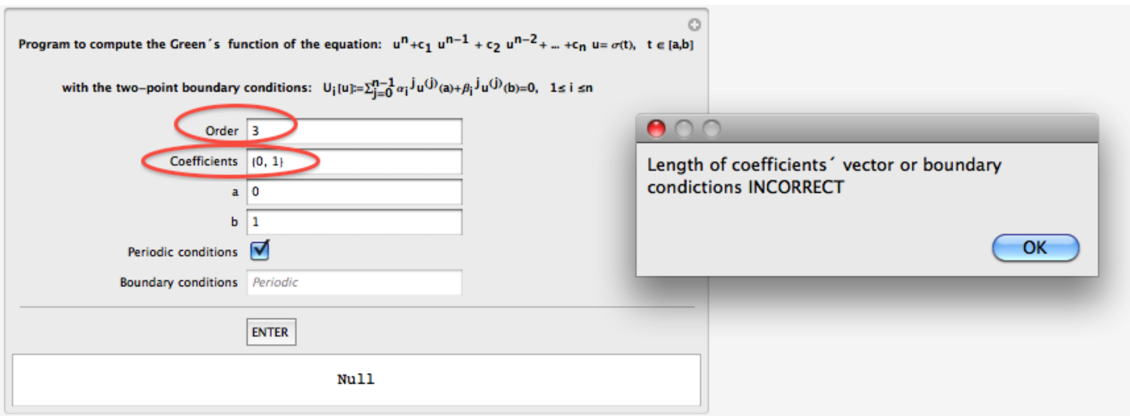


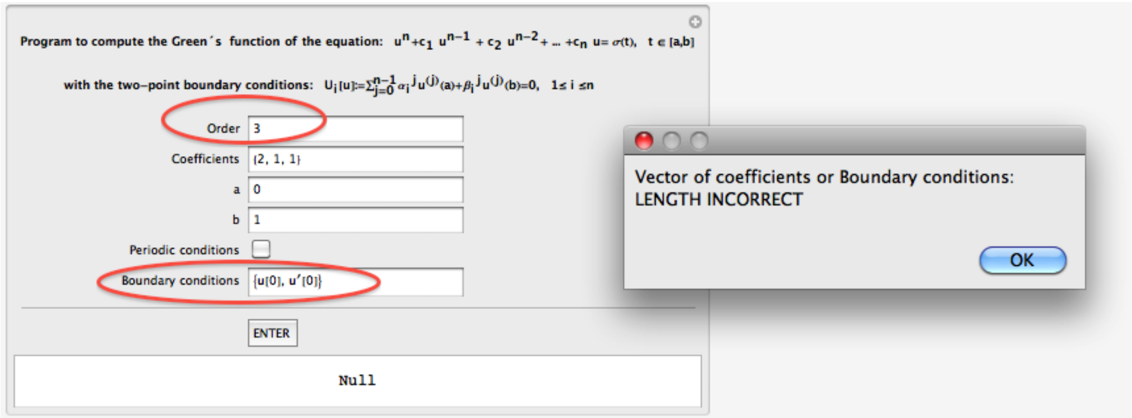
The problem with periodic boundary conditions can also be solved by entering them on the **Boundary conditions** box. In this case, the simplified expression of the solution can be obtained on a different way.



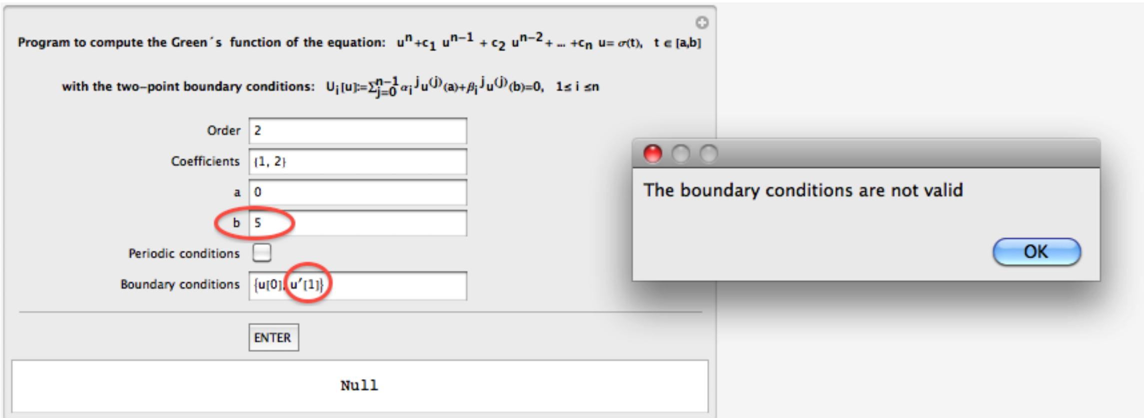
6.3. Errors

If the number of the coefficients or the boundary conditions is not the same as the order of the equation the program will warn us.

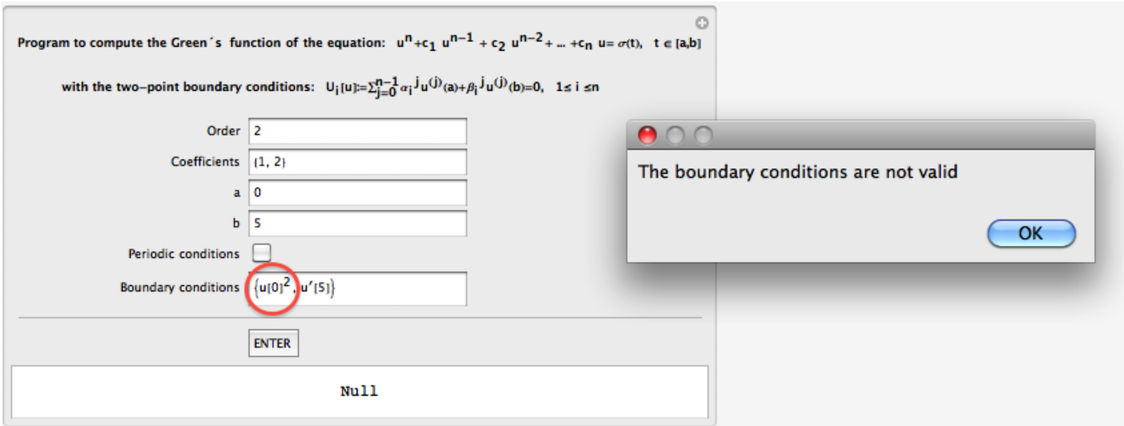




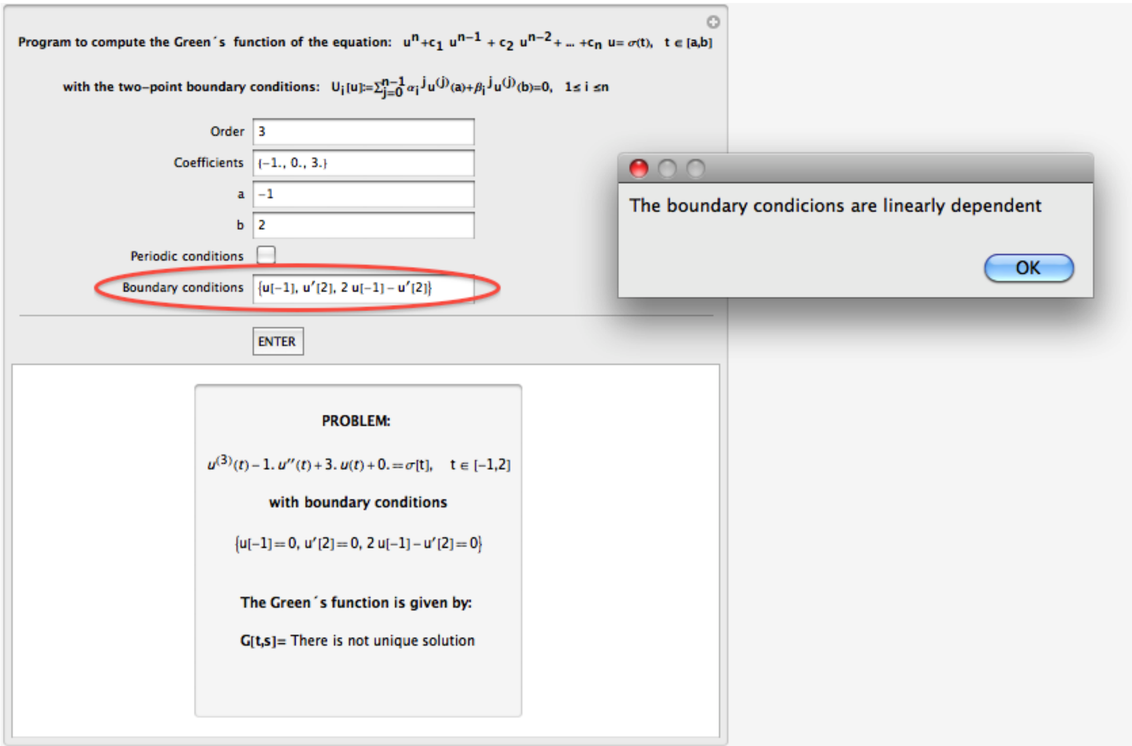
We notice again that the boundary conditions must be evaluated at the endpoints of the interval *a* and *b*. For instance, since the considered interval is [0,5], the boundary conditions {u[0],u'[1]} are not valid for the following second order problem:



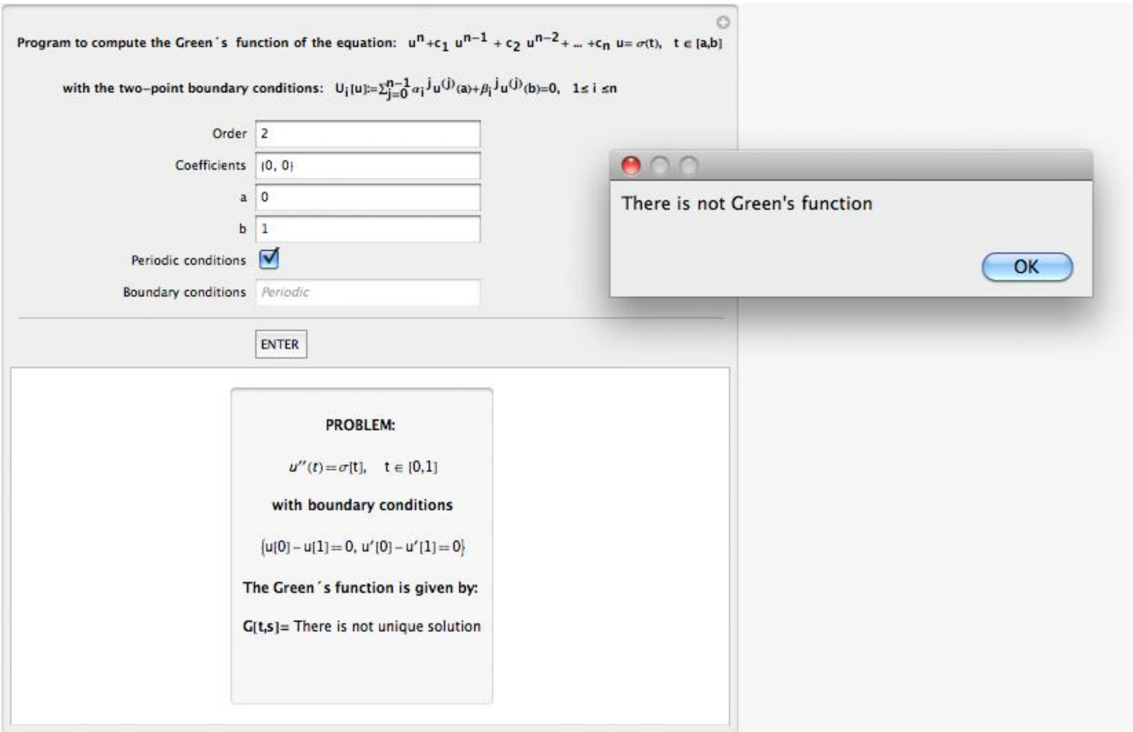
The program will also warns about other errors, for instance, if any of the boundary conditions is not linear or if it depends on a derivative bigger than or equal to the order of the equation. Notice for example that condition $u[0]^2 = 0$ is not allowed (although it is equivalent to $u[0] = 0$ that it would be valid):



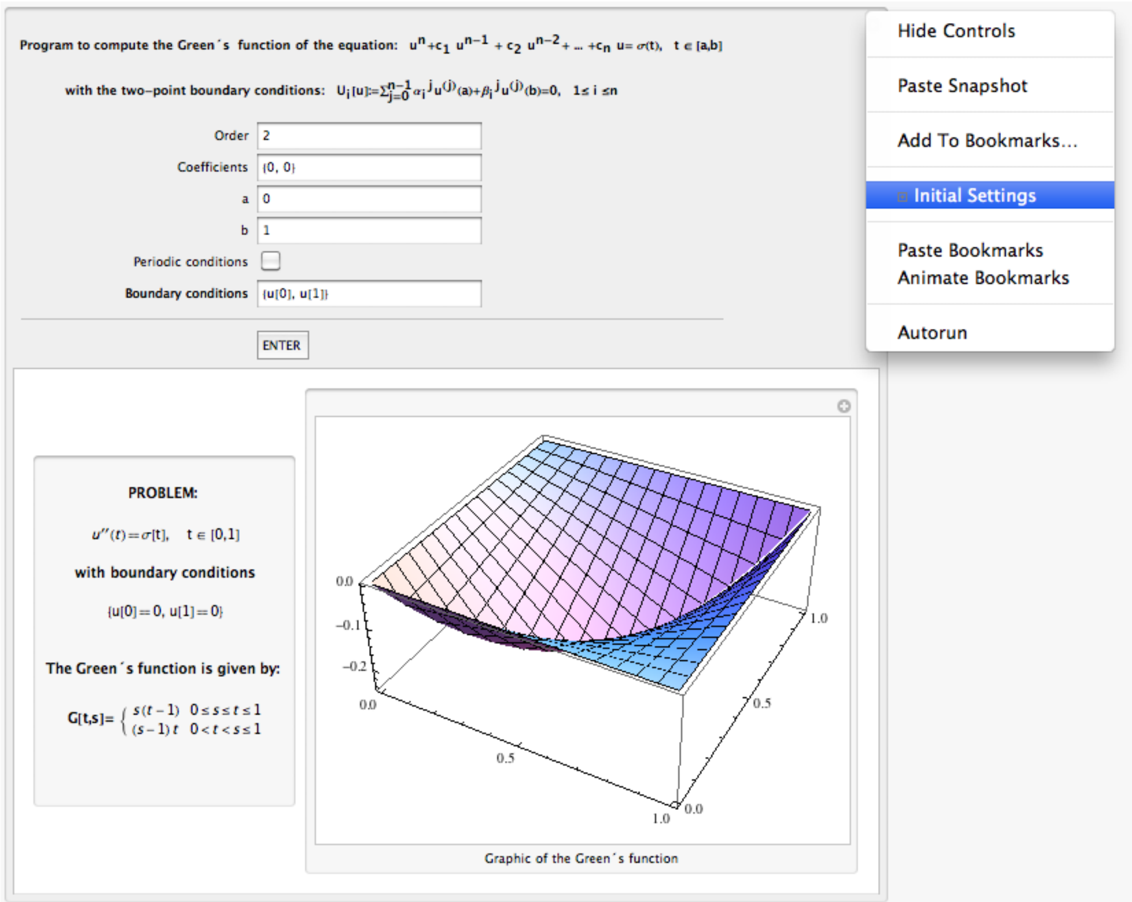
The program also check if the *n* boundary conditions are linearly independent:



The resonant problems, i. e., when the Green's function doesn't exist, are also detected by the program:



If *Mathematica,* after five minutes, is not able to calculate the Green's function for the considered problem, then an error message is shown, alerting to the user about overtime. Have been detected some examples where *Mathematica* were not able to show the expression obtained for the Green's function on the notebook. In this case the program seems like blocked. The evaluation can be aborted by using "Evaluation -> Interrupt Evaluation" on the *Mathematica* menu. After this, to restart the initial settings the symbol "+" on the upper-right corner of the program must be pressed and "Initial Settings" selected.



6.4. Global variables after the execution

The main goal of this program is to obtain the expression of the Green's function in the most standard way. Is for this that some variables of the program are global, so, after an execution, the user can work directly with them on the *Mathematica* notebook. Namely, the Green's function, G[t,s], is a global variable, in consequence if, after an execution, the user writes "G[t,s]" on a new input cell of *Mathematica,* the program gives its expression. In this way it is possible to manipulate or plot it at the convenience of the user.

Also G1[t,s] and G2[t,s], the Green's function restricted to s<t and s>t, respectively, are global variables.

Bibliography

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