BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS

FACULTY OF CIVIL ENGINEERING

György Popper

SOME CONCEPTS OF FUNCTIONAL ANALYSIS USING MATHEMATICA

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György Popper

Research Group for Computational Structural Mechanics
Hungarian Academy of Sciences-Budapest University of Technology and Economics

e-mail: gypopper@epito.bme.hu

Contents

Preface

This textbook contains the course of lectures about some concepts in functional analysis that I have delivered periodically to Ph.D. students of civil engineering at the Budapest University of Technology and Economics.

Functional analysis is a study of abstract, primarily linear, spaces resulting from a synthesis of geometry, linear algebra and mathematical analysis. Functional analysis generalizes mathematical disciplines, and its popularity originates from its geometric character: Most of the principal results in functional analysis are expressed as abstractions of intuitive geometric properties of the three-dimensional space.

The importance of getting acquainted with functional analysis lies in its frequent use in the up-to-date engineering literature. I aim to present the basics of functional analysis believed necessary to understand the mathematical theory of the finite element method, variational solution of boundary-value problems, as well as other problems of continuum mechanics.

For the purpose of this text, it is only necessary to acquire a simple understanding of the Lebesgue integral and some other concepts related to it. Hence, I have decided to avoid introducing a formalized framework for the Lebesgue measure and integration theory. However, a short introduction to the Lebesgue integration theory is given in Appendix B.

Some of the calculations of this textbook were made using the symbolic-numeric computer algebra system *Mathematica*. I found this system very useful for presenting these concepts because it does algebra and calculus computations quickly in an exact, symbolic manner.

During the preparation of this textbook, I received helpful suggestions from Professors Zsolt Gáspár, Béla Paláncz and Tibor Tarnai whom I would like to express my thanks. I would also like to thank Lajos R. Kozák for his help in preparing this manuscript for the press.

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Budapest, April, 2006. György Popper

1. Vector spaces, subspaces, linear manifolds

It is well known, that the sum of vectors in a plane and the product of a vector by real numbers results in a vector in the same plane. In other words, the set is closed for these two operations. These "space"-properties of the geometric vector space hold for some other sets too, e.g. for some function sets. Hence it is advantageous to introduce the following definition:

Linear space or **vector space** is a non empty set X of elements, often called vectors, for which two operations are defined:

1. *Addition*, that is if $x, y \in X$ then $x + y \in X$

2. M*ultiplication by scalar,* that is if $x \in X$ and α is arbitrary scalar, then $\alpha x \in X$.

These two operations are to satisfy the usual axioms:

If $x, y, z \in X$ and $\alpha, \beta \in \Re$ $(or\ C)$ [1], then

1. $x + y = y + x$ (law of commutatitvity);

2. $(x + y) + z = x + (y + z)$ (law of associativity);

3. there exists an element $\Theta \in X$ such, that $\Theta + x = x$ for any $x \in X$ (existence of the zero element);

4. for any $x \in X$, there is an element $-x \in X$ such as that $x + (-x) = \Theta$ (existence of negative elements);

5. $\alpha(x + y) = \alpha x + \alpha y$; (law of distributivity with respect to vectors);

6. $(\alpha + \beta)x = \alpha x + \beta x$; (law of distributivity with respect to scalars);

7. $(\alpha\beta)x = \alpha(\beta x)$; (law of associativity for scalar multiplication);

8. $0x = \Theta$ and $1x = x$.

If the scalars are real X is a real vector space, while if the scalars are complex X is a complex vector space.

---

[1] $\Re$ (C) denotes the set of real (complex) numbers

Examples

E.1.1. The set $\Re^n$ of all real ordered n-tuples $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$, … is a real linear space if addition is defined by $x+y=(x_1+y_1,...,x_n+y_n)$, and scalar multiplication is defined by $\alpha\, x=(\alpha\, x_1,...,\alpha\, x_n)$ with $\alpha\in\Re$. The zero vector: $\Theta=(0,...,0)$.

E.1.2. The set $C^{(0)}[a,b]$ of all (real-valued) continuous functions on a finite interval $a\le t\le b$ with addition and real number multiplication

$$(f+g)(t)=f(t)+g(t), \qquad (\alpha f)(t)=\alpha f(t),\ t\in[a,b]$$

forms a linear space. The zero vector Θ: $f(t)=0$ for all $t\in[a,b]$.

Note, that $C^{(0)}[a,b]\subset C^{(0)}(a,b)$ is true, because $f\in C^{(0)}[a,b]\Rightarrow f\in C^{(0)}(a,b)$.

More generally, $C^{(k)}[a,b]$ denotes the linear space of *k*-times continuously differentiable functions on a finite closed interval $[a,b]$. (This means the set of functions whose derivatives at least up to order k inclusive are continuous in $[a,b]$.)

For example, the function

$$f(t)=\begin{cases} t+\dfrac{t^2}{2}, & \text{if } -1\le t\le 0 \\ t-\dfrac{t^2}{2}, & \text{if } 0\le t\le 1 \end{cases}$$

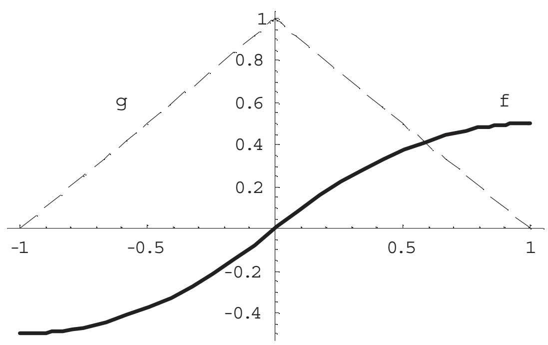
defined on the interval $-1\le t\le 1$, is once continuously differentiable but only once, i.e. $f(t)\in C^{(1)}[-1,1]$ but its first derivative

$$g(t)=\frac{df}{dt}(t)=1-|t|, -1\le t\le 1$$

belongs only to $C^{(0)}[-1,1]$, (see Fig.1.1).

Using *Mathematica*:

```
In[1]:= Clear[f, g]

In[2]:= f[t_] := Piecewise[{{t + t^2/2 , -1 ≤ t ≤ 0}, {t - t^2/2 , 0 ≤ t ≤ 1}}]
        g[t_] = D[f[t], t] // Simplify

Out[3]= 1              t == 0
        Indeterminate  t == -1 || t == 1
        1 - t          0 < t < 1
        1 + t          -1 < t < 0

In[4]:= Plot[{f[t], g[t]}, {t, -1, 1},
         PlotStyle→ {
           {Thickness[.008]}, {Dashing[{.04, .02}]}
          },
         Prolog→ {
           Text["f", {0.9, 0.6}],
           Text["g", {-0.6, 0.6}]
          }
        ]
```




```
Out[4]= - Graphics -
```

Figure1.1. Only once differentiable function.

In spite of the generality of vector spaces, it is easy to find a class of functions which does not create a linear space:

E.1.3. The class of positive functions with usual addition and number multiplication rules (introduced in **E.1.2.**) does not form a linear space. Really, if $f(t) > 0$ then $(-1) f(t) < 0$.

E.1.4. The set of functions with property

$$|f(t)|<1, \quad t\in[a,b]$$

does not form a linear space (if $f \neq \Theta$ and the number $\alpha > 0$ is chosen sufficiently large, then $|\alpha f| >1$).

However, in general: The class of *bounded* functions with usual addition and number multiplication is a linear space.

Direct product. If X and Y are two *sets*, then the *set* of all ordered pairs

$$\{(x,y): x\in X, y\in Y\}$$

is called *direct*- or *Cartesian or Descartes product* of sets X and Y and is denoted by $X\otimes Y$.

The direct product $X\otimes Y$ can be obtained using the *Mathematica* function *Outer[]*, which combines each element of the list given in its second argument with each element of the list given in its third argument, while applying the function given in the first argument to all of the possible pairs. The result is in the form of a list:

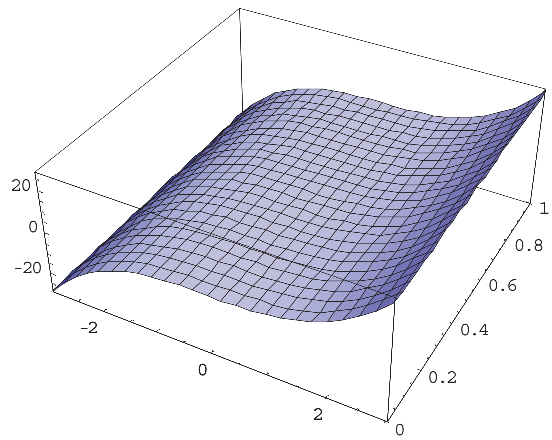
```
In[1]:= Outer[f, {a, b, c}, {1, 2}]     (* outer product *)

Out[1]= {{f[a, 1], f[a, 2]}, {f[b, 1], f[b, 2]}, {f[c, 1], f[c, 2]}}

In[2]:= Outer[List, {a, b, c}, {1, 2}]

Out[2]= {{{a, 1}, {a, 2}}, {{b, 1}, {b, 2}}, {{c, 1}, {c, 2}}}

In[3]:= Flatten[%]     (* flatten all levels in the list *)

Out[3]= {a, 1, a, 2, b, 1, b, 2, c, 1, c, 2}

In[4]:= Flatten[%%, 1]     (* flattens the topmost level in the list *)

Out[4]= {{a, 1}, {a, 2}, {b, 1}, {b, 2}, {c, 1}, {c, 2}}
```

Consequently it is reasonable to define the function:

```
In[5]:= Descartes[x_, y_] := Flatten[Outer[List, x, y], 1]
        X = {a, b, c};
        Y = {1, 2};
        Descartes[X, Y]

Out[8]= {{a, 1}, {a, 2}, {b, 1}, {b, 2}, {c, 1}, {c, 2}}
```

In *Mathematica*, the direct product can occur as an argument or definition domain of many inner functions. For example:

```
In[9]:= Plot3D[x^3 + y^2 - 1, {x, -3, 3}, {y, 0, 1}]
```




```
Out[9]= - SurfaceGraphics -
```

Figure1.2. Illustration to the use of Descartes product.

E.1.5. If X and Y are two *linear spaces*, then the direct product

$$X \otimes Y = \{(x, y) : x \in X, y \in Y\}$$

is a *linear space* with the following operations:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad x_1, x_2 \in X, \; y_1, y_2 \in Y$$

$$\alpha(x, y) = (\alpha x, \alpha y), \quad x \in X, \; y \in Y, \; \alpha \text{ scalar.}$$

Note, that the addition in vector space $X \otimes Y$ is defined using the additions of both linear spaces X and Y.

Subspaces. A nonempty subset S of a vector space X is called a *linear subspace* of X, if S itself is a linear space with respect to addition and scalar multiplication defined on X. In other terms, subspaces are closed for vector addition and multiplication by scalar.

The subspaces are linear spaces; hence all subspaces contain the *zero vector* Θ.

E.1.6. Any straight line and plane running through the origin of the three-dimensional geometric vector space, is its linear subspace.

Among all subsets of the plane shown in Figure 1.3, only the sketches 1.3.c) and 1.3.d) contain subspaces of the plane.

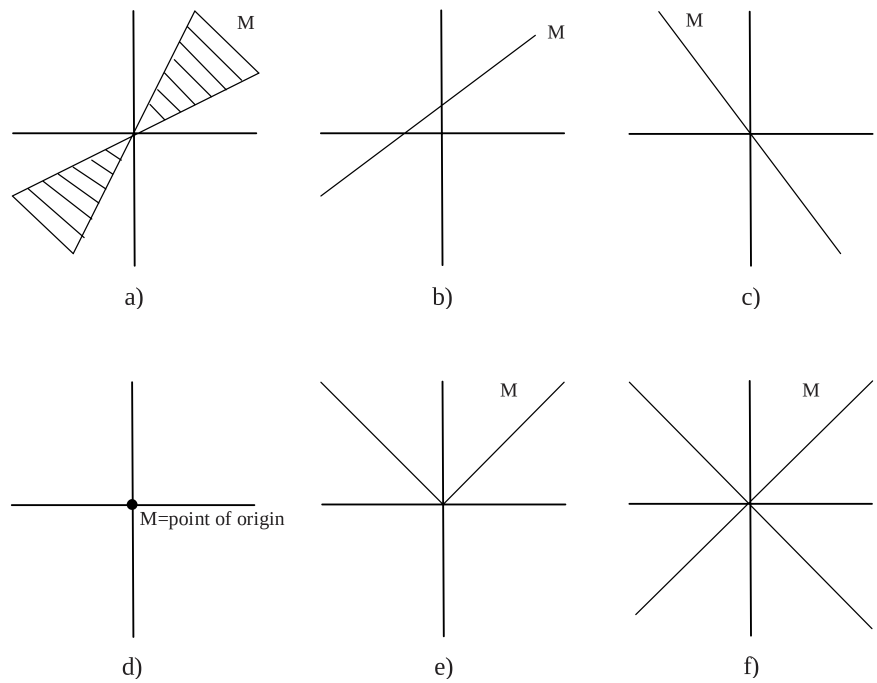


Figure1.3. Subsets and subspaces in $\Re^2$.

E.1.7. In the vector space $C^{(0)}[a,b]$ defined in the example E.1.2, the subset

$$\{f : f(a) = f(b) = 0\}$$ is a linear subspace. The subset

$$\{f : f(a) = f(b) = 1\}$$ is not a linear subspace.

E.1.8. If M and N are linear subspaces of a linear space X, then the *intersection* $M \cap N$ is also a linear subspace of X. The set

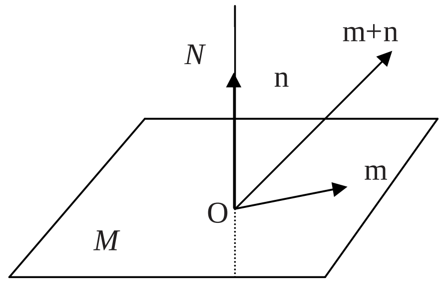
$$M + N = \{m + n : m \in M, n \in N\}$$

is a linear subspace of X, too. If

$$M + N = X \text{ and } M \cap N = \{\Theta\},$$

then both M and N is said to be the *complement* of the other with respect to X, and the vector space X is called the *direct sum* of M and N, and is denoted by $X = M \oplus N$.

For example see the sketch:



Linear manifold or **affine subspace.** Any subset of a vector space X which is of form $\{x_0 + s : s \in S\}$, where $x_0 \in X$ is a fixed vector, and S is a linear subspace, is said to be a *linear manifold* or *affine linear subspace* of X. The affine subspace $\{x_0 + s : s \in S\}$ can also be written in the form of direct sum $x_0 \oplus S$.

Linear manifolds (with $x_0 \neq \Theta$) can be considered as generalizations of straight-lines or planes which do not contain the origin.

2. Dimension, spanning sets and (algebraic) basis

The vectors $x_1, x_2, ..., x_n$ of a linear space X are said to be *linearly dependent* if there are scalars $\alpha_1, \alpha_2, ..., \alpha_n$ not all of which are zero such that

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = \Theta .$$

The vectors $x_1, x_2, ..., x_n$ are *linearly independent* if this equality holds only if all of the scalars $\alpha_1, \alpha_2, ..., \alpha_n$ are zero.

The *infinite set* of vectors $x_1, x_2, ..., x_n, ...$ is said to be linearly independent if all of its finite subsets are linearly independent.

Note, that the single vector $x \neq \Theta$ forms a linearly independent set because the only way to have $\alpha x = \Theta$ is $\alpha = 0$.

An important example of linearly independent vectors is the polynomials. For example, the powers of x, that is $\{1, x, x^2, ..., x^n\}$ form a linearly independent set. We make the proof for $n = 2$ only, because it can be extended directly (without new idea), to the general case.

Define the linear combination

$$p(x) = a_0 + a_1 x + a_2 x^2$$

and determine when $p(x) \equiv 0$. This identity implies that all derivatives of $p(x)$ are also identically equal to zero. That is,

$$\begin{aligned} p(x) &= a_0 + a_1 x + a_2 x^2 = 0, \\ p'(x) &= \qquad a_1 + 2a_2 x = 0, \\ p''(x) &= \qquad\qquad 2a_2 = 0. \end{aligned}$$

The last equation implies $a_2 = 0$, then from the second one: $a_1 = 0$, and then the first one implies $a_0 = 0$. Therefore, $p(x) \equiv 0$ if and only if $a_i = 0,\ i = 0, 1, 2, ...$

Dimension. The vector space X is called n-*dimensional* or *finite-dimensional*, if X contains n linearly independent vectors but any $n+1$ vectors of X are linearly dependent.

The linear space X is said to be *infinite-dimensional*, if for any positive integer n, n linearly independent vectors of X can be found.

Spanning set. A subset $S = \{x_1, x_2, ..., x_n, ...\}$ of a vector space X is said to *span* or *generate* X, if every $x \in X$ can be written as a linear combination of the elements of S.

Basis. A set of vectors $x_1, x_2, ..., x_n$ of a vector space X is said to be an *algebraic* or *Hamel basis* for X, if and only if

1. the set spans X, and
2. the vectors $x_1, x_2, ..., x_n$ are linearly independent.

Theorem 2.1. Let $\{x_1, x_2, ..., x_n\}$ denote a basis for a vector space X. Then every vector $x \in X$ is a unique linear combination of $x_1, x_2, ..., x_n$.

Proof: By contradiction: if $x = \alpha_1 x_1 + ... + \alpha_n x_n = \beta_1 x_1 + ... + \beta_n x_n$, then

$x - x = \Theta = (\alpha_1 - \beta_1)x_1 + ... + (\alpha_n - \beta_n)x_n$. But $x_1, x_2, ..., x_n$ are linearly independent, therefore the coefficients of $x_1, x_2, ..., x_n$ must be zero. Hence $\alpha_1 = \beta_1, ..., \alpha_n = \beta_n$.

E.2.1. Consider the vector space $C^{(0)}[a,b]$ of continuous functions defined on the finite interval $[a,b]$. Its subset consisting of polynomials of degree $\le n$ is a linear subspace in which the functions $t^k (k = 0, 1, \dots, n)$ are linearly independent. An arbitrary polynomial of degree $\le n$ can be expressed as a linear combination

$$\alpha_1 + \alpha_2 t + ... + \alpha_{n+1} t^n.$$

Therefore the vectors $t^k (k = 0,1,...,n)$ span this $(n+1)$-dimensional linear subspace and create its Hamel basis.

The linear space $C^{(0)}[a,b]$ is infinite-dimensional. Really, for any positive integer n the functions

$$x_1(t) = 1, x_2(t) = t, ..., x_{n+1}(t) = t^n$$

are linearly independent and are elements of $C^{(0)}[a,b]$.

The following example contains one of the basic concepts of the finite element method:

E.2.2. Let $C_1^{(0)}[-1,1]$ denote that linear subspace of the vector space $C^{(0)}[-1,1]$, which consists of piecewise linear polynomials on the intervals $[-1,0]$ and $[0,1]$.

This means that an arbitrary function $p(t) \in C_1^{(0)}[-1,1]$ can be written in a unique way as

$$p(t) = \begin{cases} p_1(t) \equiv a_1\, t + b_1, \text{ for } t \in [-1,0] \\ p_2(t) \equiv a_2\, t + b_2, \text{ for } t \in [0,1] \end{cases}$$

where in consequence of the continuity condition $p_1(0) = p_2(0)$ follows $b_1 = b_2$. If we express the coefficients $a_1, a_2, b_1 = b_2 = b$ using the function values $g(-1), g(0), g(1)$ at the nodal- points $-1, 0, 1$, then

$$p(t) = \begin{cases} -t\, g(-1) + (1+t)\, g(0), \text{ if } t \in [-1,0] \\ (1-t) g(0) + t\, g(1), \quad \text{ if } t \in [0,1] \end{cases}$$

This piecewise polynomial can be expressed in the form of a linear combination

$$p(t) = g(-1)\, \varphi_1(t) + g(0)\, \varphi_2(t) + g(1)\, \varphi_3(t)$$

where

$$\varphi_1(t) = \begin{cases} -t, \text{ if } t \in [-1,0] \\ 0, \text{ if } t \in [0,1] \end{cases} \quad \varphi_2(t) = \begin{cases} 1+t, \text{ if } t \in [-1,0] \\ 1-t, \text{ if } t \in [0,1] \end{cases} \quad \varphi_3(t) = \begin{cases} 0, \text{ if } t \in [-1,0] \\ t, \text{ if } t \in [0,1] \end{cases}$$

see Figure 2.1.

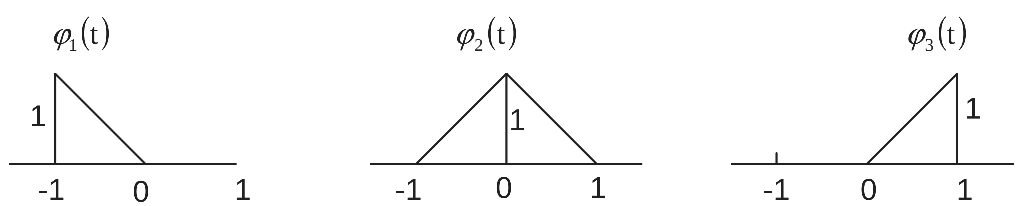


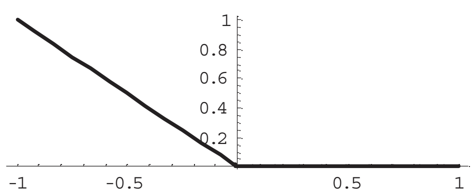
Figure 2.1. Basis in the function space $C_1^{(0)}[-1,1]$.

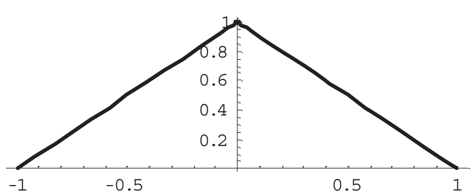
The functions $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ are linearly independent (see Figure 2.1.) and any function $p(t) \in C_1^{(0)}[-1,1]$ is a linear combination of $\varphi_1(t), \varphi_2(t), \varphi_3(t)$. Hence these functions span (generate) this 3-dimesional function space.

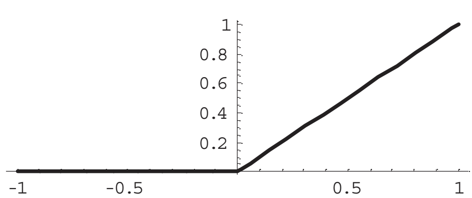
Computing the basis functions for the vector space $C_1^{(0)}[-1,1]$ can be made using *Mathematica*:

```
In[1]:= Clear[p]
        p[t_] := Piecewise[{{a1 t + b, -1 ≤ t ≤ 0}, {a2 t + b, 0 ≤ t ≤ 1}}]
```

```
In[3]:= (* g[-1],g[0],g[1] given function values of p[t_] in node points -1,0,1 *)
        s = Solve[Map[p, {-1, 0, 1}] == Map[g, {-1, 0, 1}], {a1, a2, b}] // First

Out[3]= {a1 → -g[-1] + g[0], a2 → -g[0] + g[1], b → g[0]}

In[4]:= (* substitution of the solution into p[t_]: *)
        ss = p[t] /. s

Out[4]= { g[0] + t (-g[-1] + g[0])    -1 ≤ t ≤ 0
          g[0] + t (-g[0] + g[1])     0 ≤ t ≤ 1

In[5]:= sss = {ss[[1, 1, 1]], ss[[1, 2, 1]]}

Out[5]= {g[0] + t (-g[-1] + g[0]), g[0] + t (-g[0] + g[1])}

In[6]:= (*arrange to form p[t_]:=g[-1] φ[1][t]+g[0] φ[2][t]+g[1] φ[3][t]*)
        fi = Table[Coefficient[sss, g[k - 2]], {k, 1, 3}]

Out[6]= {{-t, 0}, {1 + t, 1 - t}, {0, t}}

In[7]:= Table[
         φk[t_] = Piecewise[{{fi[[k, 1]], -1 ≤ t ≤ 0}, {fi[[k, 2]], 0 ≤ t ≤ 1}}],
         {k, 1, 3}]

Out[7]= {{ -t   -1 ≤ t ≤ 0,  { 1 + t   -1 ≤ t ≤ 0 ,  { 0   -1 ≤ t ≤ 0 }
                               1 - t    0 ≤ t ≤ 1       t    0 ≤ t ≤ 1

In[8]:= (* drawing the basis functions *)
        Do[Plot[φk[t], {t, -1, 1}, AspectRatio → 1 / 3,
           PlotStyle → Thickness[0.01]], {k, 1, 3}];
```







Training problem: Let $C_3^{(1)}[-1,1]$ denote the linear subspace of the vector space $C^{(1)}[-1,1]$ which consists of piecewise cubic polynomials on the intervals $[-1,0]$ and $[0,1]$. Similarly to example E.2.2, determine the basis of the function space $C_3^{(1)}[-1,1]$ and plot it using *Mathematica*!

A possible solution is shown in Appendix A.

E.2.3. Consider the linear space X whose elements are infinite sequences of real numbers $x=\{\xi_1,\xi_2,...\}$ with $\sum_{k=1}^{\infty}\xi_k^2<\infty$. The subset of X consisting of vectors

$$e_1=\{1,0,0,...\}$$

$$e_2=\{0,1,0,...\}$$

$$\vdots$$

$$e_k=\{0,...,0,\underset{(k)}{1},0,...\}$$

$$\vdots$$

span X which is *infinite dimensional* (really, for any positive integer k the vectors $e_1,...,e_k$ are linearly independent and are elements of X). Clearly, all of the vectors $e_1,e_2,...$ are linearly independent, too. However, they do not form a Hamel basis (because $\{e_1,e_2,...\}$ is not a finite set).

It is true, that any element of X can be written in the form

$$x=\sum_{k=1}^{\infty}\alpha_k e_k$$

but this requires to define the convergence of infinite series for vectors. Later on, the concept of bases will be extended.

E.2.4. Let S denote those infinite sequences of real numbers in which all elements except for a finite number of elements are zero. S is a subspace of the linear space X specified in example E 2.3. Any vector $x\in S$ can be written as a linear combination of a finite set $\{e_1,e_2,...,e_n\}$. Hence, these vectors form an algebraic (Hamel) basis for the subspace S.

3. Linear operator

Operator. The term *operator* is synonymous with *function*, *map* or *mapping* and *transformation.*

Given two linear spaces (sets) X and Y, an operator T from X to Y, denoted $T: X \to Y$, is a rule that assigns one and only one vector $y = T(x) \in Y$ to every vector $x \in D \subseteq X$. The subset D of X is the *domain* of T and the image of D

$$R = \{T(x): x \in D\}$$

is the *range* of T.

The operator T is considered to be given if both its domain D, co-domain Y, and the rule of transformation are given.

The operator $T: X \to Y$ is said to be *one-to-one* or *injective*, if

$$x_1 \neq x_2 \Rightarrow T(x_1) \neq T(x_2).$$

In other terms, $T: X \to Y$ is *one-to-one,* if for every $y \in R$ there is exactly one $x \in X$ such that $y = T(x)$.

The operator is said to map X *onto* Y or is called *surjective* if $R = Y$.

If T is both injective and surjective, it is called a *bijection* from X to Y.

E.3.1. Let $\Re$ denote the set of real numbers and $\Re^+$ the set of positive real numbers. Using the rule $T(x) = x^2$ define the following operators (functions):

1. $T_1 : \Re \to \Re$. This operator is not one-to-one, since both $-x$ and $+x$ are mapped into x^2. It is not surjective either, since the negative real numbers are in the co-domain $\Re$ but not in the range $\Re^+$.

2. $T_2 : \Re \to \Re^+$. This operator is not one-to-one, but it is onto.

3. $T_3 : \Re^+ \to \Re$. This operator is one-to-one, but it is not onto.

4. $T_4 : \Re^+ \to \Re^+$. This operator is bijective, i.e. it is both one-to-one and onto.

Note that although the rule $T(x) = x^2$ defining each operator $T_1, T_2, T_3,$ and T_4 is the same, the four operators are quite different.

Similarly, changes in the domain (e.g. in the boundary conditions) of a differential operator lead to a different operator (having properties different from the original one).

Linear operator. An operator $T: X \to Y$ is said to be *linear*, if the domain D of T is a linear space (that is X or a subspace of X), and if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for α, β scalars and $x, y \in X$.

For a linear operator T, we usually write Tx instead of $T(x)$.

E 3.2. Any $m \times n$ (read "m" by "n") real matrix A represents a linear operator $T: \Re^n \to \Re^m$.

The operator T is *injective* if and only if $\operatorname{rank}(A) = n$; and T is *surjective* if and only if $\operatorname{rank}(A) = m$.

(Really, if $\operatorname{rank}(A) = n$ then from the inequality $A(x - y) \neq \Theta$ follows $x - y \neq \Theta$. That is, if $x \neq y \Rightarrow Ax \neq Ay$. The case $\operatorname{rank}(A) = m$ is easy to verify by partition of A.)

E 3.3. The linear operator of *differentiation*

$$\frac{d}{dx}: C^{(1)}[a,b] \subset C^{(0)}[a,b] \to C^{(0)}[a,b]$$

is *surjective*, i.e. *onto* the range $C^{(0)}[a,b]$ (surely, every continuous function is integrable), but is not *injective* (the derivatives of $f(x)$ and of $f(x) + \text{const}$ are equal).

The *composition* of two functions, $f: X \to Y$ and $g: Y \to Z$, is defined by

$$(g \bullet f)(x) = g(f(x))$$

In *Mathematica* the composition of functions f and g can be calculated using the function Composition[]:

```
In[1]:= f[x_] := x^2; g[x_] := Sqrt[x - 1]
        h1[x_] = Composition[f, g][x]

Out[2]= -1 + x

In[3]:= h2[x_] = Composition[g, f][x]

Out[3]= Sqrt[-1 + x^2]
```

or alternatively

```
In[4]:= h1[x_] = f@g@x
Out[4]= -1 + x
```

```
In[5]:= h2[x_] = g@f@x
Out[5]= Sqrt[-1 + x^2]
```

If $T_1 : X \to Y$ and $T_2 : Y \to Z$ are linear operators, then the composition $T_2 \bullet T_1$ is a linear operator, too. Really,

$$(T_2 \bullet T_1)(\alpha x_1 + \beta x_2) = T_2(\alpha T_1(x_1) + \beta T_1(x_2)) =$$

$$= \alpha T_2(T_1(x_1)) + \beta T_2(T_1(x_2)) = \alpha (T_2 \bullet T_1)(x_1) + \beta (T_2 \bullet T_1)(x_2).$$

Null space. The null space $N(T)$ of the linear operator $T : X \to Y$ is the set

$$N(T) = \{x : x \in X,\ T x = \Theta\}.$$

The null space, also known as the *kernel* of the transformation $T : X \to Y$ is the subset of elements of X which has the zero image.

It is easy to see that the null space of T is a linear subspace of X .

(Indeed, if $x_1 \in N(T)$ and $x_2 \in N(T)$, then $T x_1 = \Theta$ and $T x_2 = \Theta$. The linearity implies $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \Theta$. That is, $\alpha x_1 + \beta x_2 \in N(T)$.

If A is an $m \times n$ matrix, then the *Mathematica* function **NullSpace[A]** gives a list of vectors that forms a basis for the null space of the matrix A.

As an example:

```
In[1]:= A = {{1, 0, 1, 2}, {0, 1, 1, 1}, {0, 0, 0, 0}}; MatrixForm[A]
Out[1]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[2]:= B = NullSpace[A]
Out[2]= {{-2, -1, 0, 1}, {-1, -1, 1, 0}}
```

Check:

```
In[3]:= A.(α B[[1]] + β B[[2]])   (* Notice the symbolic computation *)
Out[3]= {0, 0, 0}
```

If the matrix A is nonsingular, then **NullSpace[A]** gives the empty set **{ }.**

(Notice, that the *Mathematica* function NullSpace [A] is available to compute all the solutions of a homogenous set of linear algebraic equations.)

The following result is useful in the study of operator equations.

Theorem 3.1. A linear operator, $T : X \to Y$, is one-to-one if and only if its null space is trivial, $N(T) = \{\Theta\}$.

Proof: The linearity of T implies $Tx = T(x + \Theta) = Tx + T\Theta$ and hence $T\Theta = \Theta$, that is $\Theta \in N(T)$.

If T is one-to-one, then $x \neq \Theta$ implies $Tx \neq T\Theta$, that is $Tx \neq \Theta$. In other terms, if $x \neq \Theta$ then $x \notin N(T)$. Hence $N(T) = \{\Theta\}$.

Conversely, if $N(T) = \{\Theta\}$, then the equality $Tx_1 = Tx_2$ implies $T(x_1 - x_2) = \Theta$, that is, $x_1 - x_2 \in N(T)$ and since it is supposed that $N(T) = \{\Theta\}$, it follows $x_1 = x_2$. Thus, if $N(T) = \{\Theta\}$ then the operator T is one-to-one. With this the proof is completed.

Linear operators on finite-dimensional spaces. The study of continuous systems often leads to solving boundary-value problems. The solution is a vector in an infinite-dimensional space. The numerical methods approximate the solution in finite-dimensional subspaces of the infinite-dimensional space. Hence it is of great importance that:

Linear operators on finite-dimensional vector spaces can be represented by matrices.

Let X and Y be finite-dimensional linear spaces, and let $T : X \to Y$ be a linear operator. Let $\{\varphi_1, \dots, \varphi_n\}$ and $\{\psi_1, \dots, \psi_m\}$ be bases for X and Y respectively. Then $x \in X$ and $y \in Y$ can be expressed uniquely as

$$x = \sum_{i=1}^{n} \alpha_i \varphi_i, \qquad y = \sum_{j=1}^{m} \beta_j \psi_j$$

Thus for any $x \in X$, it can be written

$$y = T\,x = T\left(\sum_{i=1}^{n} \alpha_i\,\varphi_i\right) = \sum_{i=1}^{n} \alpha_i\, T\varphi_i = \sum_{j=1}^{m} \beta_j\,\psi_j\,.$$

Since $T\,\varphi_i \in Y$, that is the image of φ_i is an element of the space Y we can write

$$T\,\varphi_i = \sum_{j=1}^{m} t_{ji}\,\psi_j\,.$$

If we substitute this formula into the previous one, then because $\sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} t_{ji}\,\psi_j = \sum_{j=1}^{m}\sum_{i=1}^{n} t_{ji}\alpha_i\,\psi_j$ we get

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{n} t_{ji}\alpha_i\right)\psi_j = \sum_{j=1}^{m} \beta_j\psi_j\,,$$

and the uniqueness (see theorem 2.1.) implies

$$\sum_{i=1}^{n} t_{ji}\alpha_i = \beta_j, \qquad j = 1,\dots,m$$

or in matrix form

$$\begin{bmatrix} t_{11} & . & . & . & t_{1n} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ t_{m1} & . & . & . & t_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ . \\ . \\ . \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ . \\ . \\ . \\ \beta_m \end{bmatrix}$$

The matrix $[\,t_{ji}\,]$ is said to *represent* the linear transformation with respect to the bases $\{\varphi_1,\dots,\varphi_n\}$ and $\{\psi_1,\dots,\psi_m\}$.

Inverse operator. If $T : X \to Y$ is a one-to-one linear operator, then the linear operator $T^{-1} : R(T) \subseteq Y \to X$, which to every element $y \in R(T)$ assigns the element $x \in X$, for which $T\,x = y$, is called the *inverse* operator to the operator T.

Functional. If the operator to vectors of a linear space X assigns scalars (e.g. if $Y = \mathfrak{R}$, the set of real numbers), then the operator is called *functional* on X.

Since a functional is a special operator, the functional is said to be linear when it is a linear operator.

E.3.4. *Linear functional* is e.g. $f(x)=\int_a^b x(t)dt$

Bilinear functional. Let $X\otimes Y=\{(x,y): x\in X, y\in Y\}$ denote the direct or Cartesian product of two real linear spaces X and Y. A functional

$$b(x,y)\equiv\langle x|y\rangle$$

defined on $X\otimes Y$ is called *bilinear*, if

$$\langle \alpha x_1+x_2|y\rangle=\alpha\langle x_1|y\rangle+\langle x_2|y\rangle$$
$$\langle x|\alpha y_1+y_2\rangle=\alpha\langle x|y_1\rangle+\langle x|y_2\rangle$$

for all $x, x_1, x_2\in X$, $y, y_1, y_2\in Y$ and $\alpha\in\Re$.

The bilinear functional $\langle x|y\rangle$ (defined on $X\otimes X$) is said to be *symmetric*, if $\langle x|y\rangle=\langle y|x\rangle$, and is *positive definite* if $\langle x|x\rangle\geq 0$ and $\langle x|x\rangle=0$ if and only if $x=\Theta$.

The bilinear functional is sometimes also called *bilinear form*.

Algebraic dual space. Let X and Y be two real linear spaces. The set of all linear transformations from X to Y is itself a *linear* space and it is denoted by $L(X,Y)$.

For example, $L(\Re^n,\Re^m)$ is the set of all real $m\times n$ matrices, which is clearly a linear space. When $Y=\Re^1$, $L(X,Y)$ becomes the space of all *linear functionals* on X. This vector space is called the *algebraic dual* of X and is denoted by X^*. That is, $X^*=L(X,\Re^1)$.

Linear functionals on X are often expressed in the form of *dual-pairs*, that is as

$$f(x)\equiv\langle f\,|\,x\rangle$$

where $f\in X^*$, $x\in X$ and $\langle .|.\rangle$ is a bilinear map from $X^*\otimes X$ into $\Re^1$. That is, $\langle .|.\rangle: X^*\otimes X\to\Re^1$.

E.3.5. If X is a finite or n-dimensional vector space, then its algebraic dual X^* is a finite dimensional vector space, too.

There is a special relation between the bases of X and X^*. Let $\{x_1,...,x_n\}$ be a basis of X. Then any $x \in X$ uniquely can be written in the form

$$x = \sum_{i=1}^{n} \alpha_i x_i \ .$$

If F is a linear functional on the space X, then evidently

$$F(x) \equiv F\, x = \sum_{i=1}^{n} F\,(x_i)\,\alpha_i \ ;$$

that is, every linear functional is uniquely determined by its values in basis vectors $x_1,...,x_n$. Define the linear functionals $l_1,...,l_n$ by the formula

$$l_j(x) \equiv \langle l_j | x \rangle = \alpha_j \, .$$

We show that l_j is a linear functional. Indeed, for vectors $x = \sum_{i=1}^{n} \alpha_i x_i$, $y = \sum_{i=1}^{n} \beta_i x_i$ and for any scalars μ, λ, we get

$$l_j(\mu\, x + \lambda\, y) = \left\langle\, l_j \,\middle|\, \sum_{i=1}^{n} (\mu\,\alpha_i + \lambda\,\beta_i)\, x_i \right\rangle = \mu\,\alpha_j + \lambda\beta_j = \mu\, l_j(x) + \lambda\, l_j(y).$$

With respect to the trivial relation $x_i = 0\,x_1 + ... + 1\,x_i + ... + 0\,x_n$, it is obvious that the relationship between $\{l_j\}$ and $\{x_i\}$ is

$$l_j(x_i) \equiv \langle l_j | x_i \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} .$$

Since the basis vectors x_i are linearly independent, l_j are also linearly independent. Therefore the set $\{\, l_j \}$ form a basis for the dual space X^* and we call $\{\, l_j \}$ the *dual basis*.

Using the formula $l_j(x) \equiv \langle l_j | x \rangle = \alpha_j$, any linear functional $F(x)$ can be written in usual form of linear combination

$$F(x) = \sum_{i=1}^{n} F(x_i)\, l_i(x).$$

4. Normed spaces

Norm. Let X be a linear space. A functional $\|\cdot\|: X \to \Re^{+}$ (the non-negative real numbers) is a *norm on* X if and only if

N1. $\|x\| \geq 0$ for every $x \in X$ and $\|x\| = 0$ implies $x = \Theta$;

N2. $\|\alpha x\| = |\alpha| \, \|x\|$, $x \in X$, α - is a scalar;

N3. $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in X$ (triangle inequality).

Normed vector spaces. Let X be a linear space and $\|\,.\,\|$ be a norm on X. Then the *pair* $(X, \|\,.\,\|)$ is called *normed space* or *normed linear space*.

Distance. The function

$$d(x, y) = \|x - y\|, \quad x, y \in X$$

is a possible measure for distance and is said to be *distance* between vectors x and y.

Note that this definition of distance (induced by the norm) is a special distance (*metric*) defined in the theory of *metric spaces*. The conditions for general distance in metric spaces are:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

2. $d(x, y) = d(y, x)$;

3. $d(x, z) \leq d(x, y) + d(y, z)$.

E.4.1. With the linear space $\Re^n$ (of n-tuples of real numbers) the most frequently associated norms are:

$$\|x\|_1 = \sum_{k=1}^{n} |x_k| \qquad \text{(sum norm)},$$

$$\|x\|_2 = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2} \qquad \text{(Euclidean norm)}$$

(however, $\sum_{k=1}^{n} x_k^2$, is not a norm because it contradicts axioms N2 and N3)

$$\| x \|_\infty = \max_{k=1}^{n} | x_k | \qquad \text{(max or infinite norm).}$$

With the same vector space $\Re^n$, the different norms $\| x \|_1$, $\| x \|_2$, and $\| x \|_\infty$ define different normed spaces, usually denoted l_n^1, l_n^2 and l_n^∞.

Notice, that the norms and the corresponding normed spaces in above are special cases of the normed spaces l_n^p with norms

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}, \ 1 \le p < \infty \qquad \text{and} \qquad \|x\|_\infty = \lim_{p \to \infty} \|x\|_p .$$

In *Mathematica 5.0* , vector p-norms can be computed using the function

Norm [x,p]

where p can be omitted if $p = 2$.

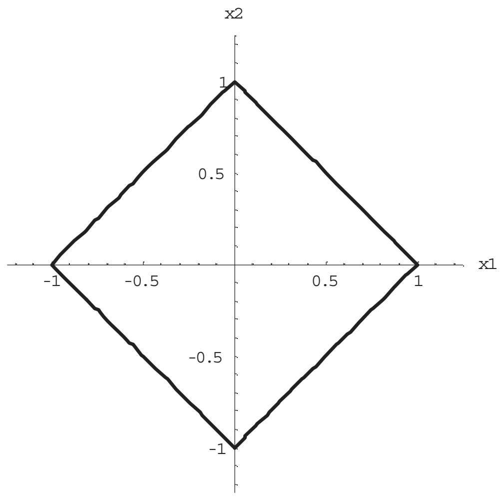
```
In[1]:= Clear["Global`*"]
        x = {a, b, c}; y = {3, -4};
        {Norm[x, ∞], Norm[x], Norm[x, 1]}

Out[3]= {Max[Abs[a], Abs[b], Abs[c]],
         √(Abs[a]^2 + Abs[b]^2 + Abs[c]^2), Abs[a] + Abs[b] + Abs[c]}

In[4]:= {Norm[y, ∞], Norm[y], Norm[y, 1]}

Out[4]= {4, 5, 7}

In[5]:= {Norm[x, p], Norm[y, p]}

Out[5]= {(Abs[a]^p + Abs[b]^p + Abs[c]^p)^(1/p), (3^p + 4^p)^(1/p)}
```

Using *Mathematica* it is easy to sketch the unit balls

$$S_p = \{ x \in \Re^2 : \|x\|_p \le 1 \}.$$

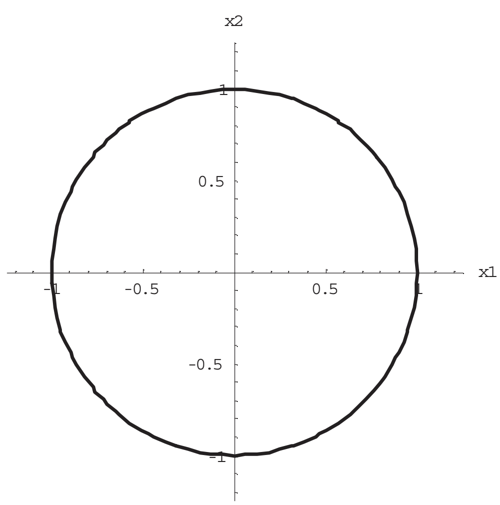
An example for $p = 1, 2, \infty$:

In[6]:=
```
<< Graphics`ImplicitPlot`
ImplicitPlot[Norm [{x1, x2}, 1] == 1, {x1, -1.2, 1.2}, {x2, -1.2, 1.2},
 AxesOrigin → {0, 0}, AxesLabel → {"x1", "x2"}, PlotStyle → {Thickness[0.008]}]
```



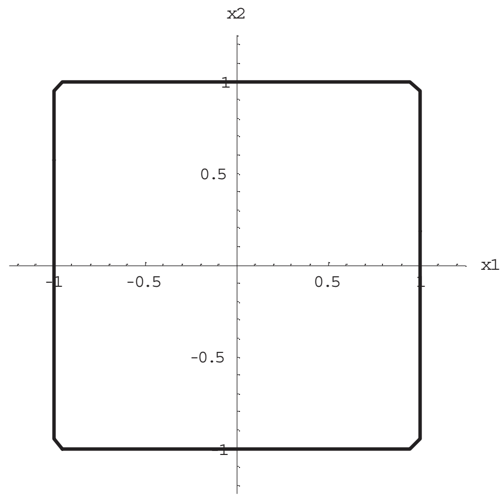
Out[7]= - ContourGraphics -

In[8]:=
```
ImplicitPlot[Norm [{x1, x2}] == 1, {x1, -1.2, 1.2}, {x2, -1.2, 1.2},
 AxesOrigin → {0, 0}, AxesLabel → {"x1", "x2"}, PlotStyle → {Thickness[0.008]}]
```



Out[8]= - ContourGraphics -

```
In[9]:= ImplicitPlot[Norm[{x1, x2}, ∞] == 1, {x1, -1.2, 1.2}, {x2, -1.2, 1.2},
         AxesOrigin → {0, 0}, AxesLabel → {"x1", "x2"}, PlotStyle → {Thickness[0.008]}]
```




```
Out[9]= - ContourGraphics -
```

E.4.2. With the space $C^{(0)}[a,b]$ of continuous functions on the interval $[a,b]$ different normed spaces can be established. The norms

$$\| x \|_1 = \int_a^b | x(t) | \, dt , \qquad \| x \|_2 = \left(\int_a^b | x(t) |^2 dt \right)^{1/2} \quad \text{and} \quad \| x \|_\infty = \max_{a \le t \le b} | x(t) |$$

define normed spaces $L_0[a,b]$, $L_0^2[a,b]$ and $C[a,b]$, respectively.

Note, that the norms in example E.4.1 are the discrete analogues of the norms in E.4.2.

In *Mathematica* e.g. the norm of the space $L_0^2[a,b]$ can be defined as follows:

```
In[1]:= Func2Norm[f_, a_, b_] := Sqrt[∫_a^b Abs[f[t]]^2 ⅆt]

In[2]:= g[x_] := x^3

In[3]:= Func2Norm[g, -1, 1]

Out[3]= Sqrt[2/7]

In[4]:= % // N

Out[4]= 0.534522
```

E.4.3. The vector space $C^{(2)}[a,b]$ of twice continuously differentiable functions on the finite interval $[a,b]$ with the norm

$$\| x \| = \left(\int_a^b | x(t) |^2 dt + \int_a^b \left| \frac{dx(t)}{dt} \right|^2 dt + \int_a^b \left| \frac{d^2 x(t)}{dt^2} \right|^2 dt \right)^{1/2}$$

$$= \left(\sum_{k=0}^{2} \int_a^b \left| \frac{d^k x(t)}{dt^k} \right|^2 dt \right)^{1/2}$$

is also a normed space.

The same linear space $C^{(2)}[a,b]$ with the norm

$$\| x \| = \max_{a \le t \le b} | x(t) | + \max_{a \le t \le b} \left| \frac{dx(t)}{dt} \right| + \max_{a \le t \le b} \left| \frac{d^2 x(t)}{dt^2} \right|$$

is another normed space.

E.4.4. The real valued function

$$\left(\int_a^b \left| \frac{dx(t)}{dt} \right|^2 dt \right)^{1/2}$$

is not a norm in the linear space $C^{(1)}[a,b]$, because it contradicts axiom N1. Clearly, from $\int_a^b | x'(t) |^2 dt = 0$ it follows only $x'(t) \equiv 0$ (and not $x(t) \equiv 0$) everywhere in $[a,b]$.

5. Convergence, complete spaces

Iterative methods are the only choice for the numerical solution of operator equations. For this reason, it is necessary to extend the concept of convergence and limit of sequences of numbers to sequences of vectors in linear spaces.

Convergence. A sequence of elements $\{x_n\}$ in *normed space* X is said to be *convergent*, if there exists $x^* \in X$, so that the sequence of numbers $\|x_n - x^*\|$ converge to zero. We refer to x^* as the *limit* of the sequence $\{x_n\}$ and write

$$x_n \to x^* \text{ as } n \to \infty \qquad \text{or} \qquad \lim_{n\to\infty} x_n = x^*.$$

We emphasize that in this definition, the condition $x^* \in X$ is of fundamental importance. Surely, the sequence of rational numbers $\{1, 1.4, 1.41, 1.414, 1.4142, \ldots\}$ in the normed space of real numbers $(\Re, |\cdot|)$ converges to $\sqrt{2}$, but in the normed space of rational numbers $(Rac, |\cdot|)$ this sequence does not converge.

Cauchy sequence. A sequence $\{x_n\}$ in a normed space is called *Cauchy sequence*, if

$$\|x_n - x_m\| \to 0, \qquad \text{as} \qquad n, m \to \infty.$$

Theorem 5.1 In normed spaces every convergent sequence is a Cauchy sequence.

Proof: This follows immediately from the triangular inequality

$$\| x_n - x_m \| = \left\| \left(x_n - x^*\right) + \left(x^* - x_m\right)\right\| \le \| x_n - x^* \| + \| x_m - x^* \|$$

supposing $x_n \to x^*$, $x_m \to x^*$ as $m, n \to \infty$. (From the norm-axiom N2 it follows $\| x^* - x_m \| = \| (-1)\left(x_m - x^*\right)\| = |-1| \, \| x_m - x^* \|$).

In the finite-dimensional space $\Re^n$, the converse of this statement is also true; any Cauchy sequence is convergent. However, in general infinite-dimensional spaces, a Cauchy sequence may fail to converge.

Complete space. A normed space X is said to be *complete* if every Cauchy sequence in X has a limit (in X).

Banach space. A *complete normed linear space* is called *Banach space*.

Note, that not only normed spaces but other metric spaces may be complete too, but Banach space may be only a normed space.

E.5.1. Consider the linear space X whose elements are infinite sequences of real numbers $x = \{\xi_1, \xi_2, ...\}$ with $\sum_{k=1}^{\infty} \xi_k^2 < \infty$. The norm in X can be defined by

$$\| x \| = \sqrt{\sum_{k=1}^{\infty} \xi_k^2}$$

and this (infinite-dimensional) *normed* vector space is usually denoted by ℓ^2.

Let ℓ_0^2 denote those infinite sequences of ℓ^2 in which all elements are zero, except a finite numbers of elements. It is easy to see, that the sequence

$$x_n = \{ 2^{-1},\ 2^{-2},\ ...,\ 2^{-n},\ 0,\ 0,\ ... \} \in \ell_0^2$$

does not have a limit, because the sequence

$$x^* = \{ 2^{-1},\ 2^{-2},\ ...,\ 2^{-n},\ 2^{-(n+1)},\ ... \} \in \ell^2$$

does not belong to ℓ_0^2.

The completion of ℓ_0^2 is ℓ^2.

E.5.2. With the vector space $C^{(0)}[0,1]$ of continuous functions on the closed interval $[0,1]$, different *normed* spaces can be defined.

The normed space $L_0[0,1]$ defined by the *integral norm*

$$\| x \|_1 = \int_0^1 | x(t) | \, dt$$

is *not complete*, but the normed space $C[0,1]$, defined by the *maximum* (*infinite*) *norm*

$$\| x \|_\infty = \max_{0 \le t \le 1} | x(t) |$$

is *complete*, that is, a *Banach* space.

To show, that the linear space $C^{(0)}[0,1]$ of continuous functions with respect to integral norm is not complete, it is enough to find a Cauchy sequence which does not converge in $L_0[0,1]$.

Therefore consider the sequence of continuous functions on interval $[0,1]$ given by

$$x_n(t) = \begin{cases} 0, & \text{if} \quad 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} + 1, & \text{if} \quad \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1, & \text{if} \quad t \ge \frac{1}{2} \end{cases}$$

$(n \ge 2)$ and illustrated in Figure 5.1.

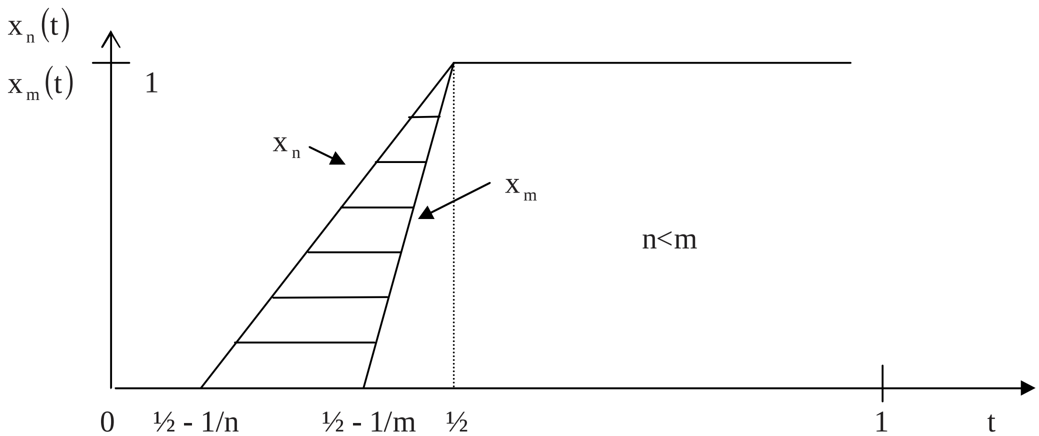


Figure 5.1. Sequence of continuous functions.

Figure 5.1 illustrates geometrically (area of lined triangle), that

$$\| x_n - x_m \|_1 = \int_0^1 | x_n(t) - x_m(t) | \, dt = \frac{1}{2} \left| \left(\frac{1}{2} - \frac{1}{m} \right) - \left(\frac{1}{2} - \frac{1}{n} \right) \right| .1 \ = \ \frac{1}{2} \left| \frac{1}{n} - \frac{1}{m} \right|,$$

and if $m \to \infty$, $n \to \infty$, then $\| x_n - x_m \|_1 \to 0$, that is, we have proved that the sequence $\{x_n(t)\}$ in the norm $\| . \|_1$ is a Cauchy sequence.

However, no continuous function $x^*(t)$ exists as a limit of the sequence $\{x_n(t)\}$.

To show it, consider the step function

$$y^*(t) = \begin{cases} 0, & \text{if } 0 \le t < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

It is easy to verify, that $\| x_n - y^* \|_1 \to 0$ as $n \to \infty$, that is, the sequence $\{x_n\}$ converges to the discontinuous function y^*. Indeed,

$\| x_n - y^* \|_1 = \frac{1}{2}\left[\frac{1}{2} - \left(\frac{1}{2} - \frac{1}{n}\right)\right].1 = \frac{1}{2n} \to 0,$ as $n \to \infty$. Moreover, the sequence $\{x_n\}$ cannot converge also to an other continuous function $x^* \ne y^*$, because if $\| x_n - x^* \|_1 \to 0$ as $n \to \infty$, then the inequality

$$\| x^* - y^* \|_1 = \| (x^* - x_n) + (x_n - y^*) \|_1 \le \| x_n - x^* \|_1 + \| x_n - y^* \|_1$$

implies $x^* = y^*$, being a contradiction.

Hence the space $L_0[0,1]$ of continuous functions with respect to the integral norm $\|.\|_1$ is not complete.

The fact, that the space $C[0,1]$ of continuous functions with respect to the maximum norm $\|.\|_\infty$ is complete, follows immediately from the theorem known in elementary analysis:

"The limit of every uniformly convergent sequence of continuous functions is a continuous function."

(In more detail we demonstrate that the normed space $C[0,1]$ is complete. For any fixed point $t_0 \in [0,1]$, a Cauchy sequence $\{x_n(t)\}$ in $C[0,1]$ yields a Cauchy sequence of real numbers: $|x_n(t_0) - x_m(t_0)| \le \max_{t \in [0,1]} |x_n(t) - x_m(t)| \to 0$, as $m, n \to \infty$. Consequently, for every $t \in [0,1]$ point there exists a real number $x(t)$ to which $\{x_n(t)\}$ converges. This pointwise convergence holds for any t in $[0,1]$; that is, Cauchy sequences $\{x_n(t)\}$ converge uniformly in $[0,1]$. Thus, there is a limit-function $x(t)$ for which

$$\| x_n(t) - x(t) \|_\infty = \max_{t \in [0,1]} | x_n(t) - x(t) | \to 0, \quad as \ n \to \infty .$$

It remains to be shown that $x(t)$ is continuous. Let $t \in [0,1]$ and let $\{t_m\}$ be a sequence of points in $[0,1]$ which converges to point t as $m \to \infty$. Clearly, for any $n \geq 1$

$$|x(t) - x(t_m)| \leq |x(t) - x_n(t_m)| + |x_n(t_m) - x(t_m)|$$

where $\{x_n\}$ is a Cauchy sequence in $C[0,1]$. Since each x_n is continuous, $x_n(t_m) \to x_n(t)$ as $m \to \infty$. Since $\{x_n(t)\}$ is a Cauchy sequence, already we have shown, that $x_n(t) \to x(t)$ as $n \to \infty$ and thus $x_n(t_m) \to x(t)$ as $n, m \to 0$. Likewise $x_n(t_m) \to x(t_m)$ as $n \to \infty$.

Hence, the previous inequality implies

$$|x(t) - x(t_m)| \to 0$$

that is, $x(t_m) \to x(t)$ if $t_m \to t$. Thus, the function $x(t)$ is continuous.)

We note, that convergence in maximum norm implies pointwise convergence, and that the sequence of continuous functions illustrated in Figure 5.1 with respect to the maximum norm is not even a Cauchy sequence. From the figure ($n < m$) it is clear, that

$$\max_{0 \leq t \leq 1} |x_n(t) - x_m(t)| = \left| x_n\left(\frac{1}{2} - \frac{1}{m}\right) \right| = n\left(\frac{1}{2} - \frac{1}{m}\right) - \frac{n}{2} + 1 = 1 - \frac{n}{m}$$

and the limit of this sequence is indeterminate. Indeed, using *Mathematica* we get

```
In[1]:= 1 - n/m /. {n → ∞, m → ∞}

— ∞::indet : Indeterminate expression 0 ∞ encountered . More…

Out[1]= Indeterminate
```

(Now we could not use the inner function *Limit[]*, because it can only work with one variable.)

Completion. Every incomplete normed space can be completed by adding the limit points of all Cauchy sequences in the space. If X is an incomplete normed space, one can construct a new space $\overline{X}$ that is complete and contains X as a subspace. The space $\overline{X}$ is called the *completion* of X.

In general the only, but not a simple problem is to find out what the limit elements will be. Remember, that the normed space $(\Re, |\cdot|)$, i.e. the set of real numbers with the usual norm (i.e. absolute value) form complete normed space.

Similarly, the normed space $(\Re^n, \|\cdot\|_2)$ is a Banach space. However, the rational numbers with the absolute value norm is not a Banach space.

Compact set. A subset S of a normed space X is said to be *compact*, if every *infinite* sequence $\{x_n : x_n \in S\}$contains a convergent subsequence, (that converges to a vector $x^* \in S$).

If S is a set for which its completion $\overline{S}$ is compact, we say S is *precompact.*

The normed space $(\Re, |\cdot|)$ is not compact. Indeed, $\{0,1,2,...\}$ contains no convergent subsequence.

Recall, that a set $S = \{x_1, x_2, ..., x_n, ...\}$ is said to be *bounded*, if there is a number $K > 0$ such that $\|x_n\| < K$ for all n. The subset S is *closed*, if $x_n \in S$ and $x_n \to x^*$ implies $x^* \in S$.

Theorem 5.2 (Heine-Borel). Let X be a finite-dimensional normed linear space, and let S be a subset of X. Then S is compact if and only if S is both *closed* and *bounded*.

However, in an arbitrary Banach space the statement of this theorem is not true.

Consider the normed space ℓ^2 introduced in example E 5.1, which consists of infinite sequences of real numbers $x = \{\xi_1, \xi_2, ...\}$ with norm

$$\|x\| = \sqrt{\sum_{k=1}^{\infty} \xi_k^2}\ .$$

In ℓ^2 consider the *bounded*, *closed* ball S defined by $\|x\| \le 1$. This ball is not compact. Indeed, consider the sequence (of sequences)

$$e_1 = \{1,0,0,...\},\ e_2 = \{0,1,0,...\},\ ...$$

For $m \neq n$ we have $\| e_n - e_m \| = \sqrt{2}$. Hence the sequence $\{e_k : e_k \in S\}$ and every subsequence of S does not contain a Cauchy subsequence and so is not convergent.

However, the compact sets are bounded and closed.

6. Continuous and bounded linear operator

Let X and Y be two normed spaces. The convergence of sequences in X and Y is understood with respect to the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ associated with the spaces X and Y, respectively.

Continuous operator. An operator $T : X \to Y$ (linear or nonlinear) is said to be *continuous* at a *point* $x^* \in X$, if for every sequence $\{x_n : x_n \in X\}$ that converges to x^* the sequence $T(x_n) \to T(x^*)$ as $n \to \infty$.

If $T : X \to Y$ is continuous at every point of its domain $D \subseteq X$ we simply say that T is continuous on D.

Theorem 6.1. An operator $T : X \to Y$ is *continuous* at a *point* $x^* \in X$, if and only if for any $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ so that

$$\text{if} \quad \| x - x^* \|_X < \delta(\varepsilon), \quad \text{then} \quad \| T(x) - T(x^*) \|_Y < \varepsilon .$$

We want to emphasize, that the continuity of an operator depends on the used norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.

Theorem 6.2. If a linear operator $T : X \to Y$ is continuous at the point $x = \Theta$, then T is continuous at all points of X.

Proof: If x^* is an arbitrary vector and $x_n \to x^*$, as $n \to \infty$, then $x_n - x^* \to \Theta$ and so by the condition of the theorem we have $T(x_n - x^*) \to T\,\Theta$. The linearity $T(x_n - x^*) = Tx_n - Tx^*$ and the equality $Tx = T(x + \Theta)$ implies $T\,\Theta = \Theta$. Hence $Tx_n \to Tx^*$, so that indeed T is continuous at an arbitrary point x^*.

Bounded linear operator. A *linear* operator $T : X \to Y$ is said to be bounded (above), if there is a constant $K > 0$ so that

$$\| T\, x \|_Y \le K \| x \|_X , \quad x \in X .$$

The number K is called a *bound* of the operator T.

In view of this definition, a linear functional $F : D_F \subseteq X \to \Re$ is bounded on its domain D_F, if there is a number $K > 0$ so that

$$| F(x) | \le K \, \|x\|, \quad x \in D_F .$$

Theorem 6.3. (The equivalence of boundedness and continuity).

A *linear* operator T is continuous if and only if it is bounded.

Proof: If T is a bounded linear operator, then

$$\|Tx_n - Tx^*\|_Y = \|T(x_n - x^*)\|_Y \le K \| x_n - x^*\|_X$$

is valid for any $x_n, x^* \in X$. Therefore, T is continuous in X.

Conversely, if T is not a bounded linear operator, we shall show that T is not continuous at the point Θ. Because T is not bounded, there exist such a bounded sequence $\{x_n : x_n \in X\}$, that $\| T x_n \|_Y \to \infty$. Without loss of generality we may suppose $Tx_n \ne \Theta$ for any positive integer n. Define the sequence

$$\tilde{x}_n = \frac{x_n}{\|T x_n\|_Y} .$$

It is obvious that $\tilde{x}_n \to \Theta$, but $\| T \tilde{x}_n\|_Y = 1$, so T is not continuous.

E.6.1. In the normed space $C[a,b]$ the integral operator

$$T\ f(t) = \int_a^t f(\tau) d\tau, \quad t \in [a,b]$$

is continuous. Clearly,

$$|T\ f(t)| = \left| \int_a^t f(\tau)\ d\tau \right| \le \int_a^t | f(\tau)|\ d\tau \le \int_a^b |f(\tau)| d\tau \le (b-a) \max | f(t)|, \quad t \in [a,b]$$

and so

$$\| T\ f \|_\infty = \max_{a \le t \le b} \left| \int_a^t f(\tau)\ d\tau \right| \le (b-a) \| f \|_\infty$$

This means, that T is bounded and hence continuous too.

E.6.2. Let D denote the subspace of the normed space $C[a,b]$, which consists of differentiable functions. Then the differential operator

$$L\ f(t) = \frac{d}{dt} f(t), \qquad f \in D$$

is *not* continuous in $C[a,b]$. To show it, it is enough to find only one convergent sequence $\{g_n : g_n \in D\}$ so, that the sequence $\{L\, g_n,\ n=1,2,...\}$ does not converge. If $g_n(t)=\dfrac{1}{n}\sin(nt)$, then $g_n \in D$ and $g_n \to \Theta$ as $n\to\infty$, however the sequence $L\, g_n(t)=\cos(nt)$ does not have a limit. Thus the operator L is not continuous (and by the theorem 6.3 it is also *unbounded)*.

We note, that by appropriate selection of norm, the operator of differentiation can be made continuous (and hence bounded too). For instance, let $C^{new}[a,b]$ denote the normed space of functions from the linear space $C^{(1)}[a,b]$ with the norm defined by

$$\| f\|_{new} = \max_{a\le t\le b} |f(t)| + \max_{a\le t\le b}\left|\frac{df(t)}{dt}\right| = \| f\|_\infty + \| f'\|_\infty .$$

The operator

$$L = \frac{d}{dt} : C^{new}[a,b] \to C[a,b]$$

i.e. the differentiation using the norm $\| f\|_{new}$ is continuous. Really, if $\| f_n\|_{new} \to 0$, that is, if $\| f_n\|_\infty + \| f_n'\|_\infty \to 0$, then $\| f_n'\|_\infty = \| L\, f_n\|_\infty \to 0$ as $n\to\infty$.

Norm of linear operator. Let X, Y be normed spaces. The smallest bound of the continuous linear operator $T : X \to Y$, which is the smallest number K for which

$$\| T\, x\|_Y \le K \| x\|_X$$

is called the *norm* of T and is denoted by $\| T\|$.

It follows that $\| T\, x\|_Y \le \| T\| \| x\|_X$.

The definition of the norm of continuous linear operator T also can be written as

$$\| T\| = \sup_{x\ne\Theta} \frac{\| T\, x\|_Y}{\| x\|_X} = \sup_{\| x\|_X = 1} \| T\, x\|_Y .$$

(The second form follows from the first one, if the vector $x \ne \Theta$ is written as $x = \| x\|_X\, \xi,\ \| \xi\|_X = 1$.)

If $T = F$ is a continuous linear *functional*, then clearly

$$\| F \| = \sup_{x \neq \Theta} \frac{| F(x) |}{\| x \|} = \sup_{\| x \| = 1} | F \ (x) | \ .$$

It can be shown, that $\| T \|$ satisfies the norm axioms.

Topological dual space. Let X be a normed space. The set of all *continuous* linear *functionals*

$$f : X \to \Re$$

is a *normed* space, denoted by X', and called the (topological) *dual* of X .

Let $f \in X'$ and express the functional f as a *duality pair*

$$f(x) = \langle f \mid x \rangle, \ f \in X', \ x \in X \ ,$$

where the symbol $\langle \cdot | \cdot \rangle$ denotes a bilinear map from the space $X' \otimes X$ into $\Re$.

If we substitute $f = \langle f \mid x \rangle$ into the formula for the norm of continuous linear functional above, we get the norm

$$\| f \|_{X'} = \sup_{x \neq \Theta} \frac{|\langle f \mid x \rangle|}{\| x \|_X} .$$

of the dual normed space X'.

Compact operator. A linear operator T is *compact* if it transforms bounded sets into compact sets.

7. Dense sets, separable spaces

Dense set. Let X be a *normed* space. A set $S \subset X$ is said to be *dense* in X if for every element $x \in X$ there is a sequence $\{s_n : s_n \in S\}$ such that $s_n \to x$ as $n \to \infty$.

In other words, any element in X can be reached as a limit of a subsequence selected from the elements of the dense set S.

Theorem 7.1. Let X be a normed space. The set $S \subset X$ is dense in X if and only if for every $\varepsilon > 0$ and every $x \in X$ there is an element $s \in S$ such, that

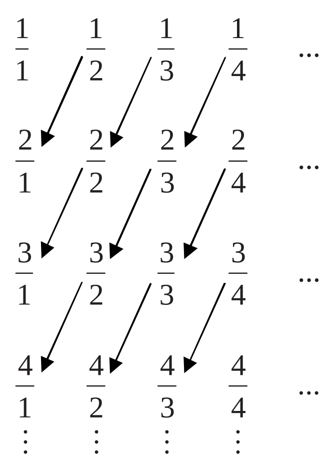
$$\| x - s \| < \varepsilon .$$

In other words, the set S is dense in the normed space X if and only if any $x \in X$ can be *approximated* (with arbitrary precision) by elements of S.

Separable spaces. The *normed* space X is said to be *separable*, if it contains a countable dense subset.

In other words, the separability of X means, that there exists a *sequence* $\{x_n : x_n \in X\}$ such that every $x \in X$ is either an element of this sequence, or x is a limit of a subsequence of $\{x_n : x_n \in X\}$.

We note, that the rational numbers, which are dense in (by absolute value) normed space of real numbers, are countable (see the diagram below). Hence the real numbers create a separable space.[1]



[1] Note, that Georg Cantor (1845-1918) has proved, that the set of real numbers on the interval [0,1] is not countable. (We say that the points of interval [0,1] create a *continuum*.)

Recall, that a subset $S = \{x_1, x_2, ...\}$ of a vector space X is said to *span* or *generate* X, if every $x \in X$ can be written as a linear combination of the vectors of S.

Spanning set. A subset S of a *normed* space X is said to *span* or *generate* X, if the set of all linear combinations of the vectors of S is *dense* in X.

This means, that every $x \in X$ either is element of S, or is a linear combination of vectors of S, or is a limit of a sequence of such linear combinations.

Theorem 7.2. If the normed space X is *separable*, then it contains a countable subset S which spans X.

The dense sequence $\{x_n : x_n \in X\}$ is the desired subset S.

The converse of this statement is also true.

Theorem 7.3. If the spanning set S of the normed space X is countable, then X is separable.

Theorem 7.4. (*Weierstrass approximation theorem*)

Let $f \in C[a,b]$, and let $\varepsilon > 0$. Then there exists a polynomial $p(x)$ for which

$$\| f - p \|_\infty \le \varepsilon .$$

E.7.1. According to the Weierstrass approximation theorem, any continuous function on a finite interval $[a,b]$ can be established as a limit of a *uniformly* convergent sequence of polynomials. This means, that polynomials form a *dense* subspace in the normed space $C[a,b]$. Every polynomial can be written as a linear combination of the power functions $1, t, t^2, ...$; hence the sequence $\{t^k;\ k = 0,1,2, ...\}$ *spans* the space $C[a,b]$, and by the theorem 7.3 it is *separable*, too.

8. Inner product, Hilbert space

Inner product. Let X be a *real linear* space. A function $\langle \cdot | \cdot \rangle : X \times X \to \Re$, defined for each pair $x, y \in X$, is called an *inner product* (or *scalar product*) on X if and only if it satisfies the following axioms:

S1. $\langle x | y \rangle = \langle y | x \rangle$,

S2. $\langle x + y | z \rangle = \langle x | z \rangle + \langle y | z \rangle; \; z \in X$,

S3. $\langle \alpha x | y \rangle = \alpha \langle x | y \rangle$; α-real number,

S4. $\langle x | x \rangle \geq 0 \, \acute{e}s \langle x | x \rangle = 0$ if and only if, ha $x = \Theta$.

This definition is a generalization of the scalar product of geometric vectors.

We note, that if X denote *complex* linear space, then the axiom S1 is replaced by $\langle x | y \rangle = \overline{\langle y | x \rangle}$, (upper bar denotes complex conjugate).

Observe, that the inner product is a *bilinear functional* which is *positive definite* i.e. *symmetric,* too.

Inner product space. A real linear space with an inner product is called inner product space or *pre-Hilbert space* (in *finite* dimensional cases also *Euclidean* space).

E.8.1. In the linear space $\Re^n$ of real n-tuples $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$

$$\langle x | y \rangle = x_1 y_1 + ... + x_n y_n$$

is the usual *euclidean inner product.*

E.8.2. In the real vector space $C^{(1)}[a,b]$ of all continuous functions with continuous first derivative, the expressions

$$\langle x | y \rangle = \int_a^b x(t) y(t) dt$$

and

$$\langle x | y \rangle = \int_a^b x(t) y(t) \, dt + \int_a^b \frac{dx(t)}{dt} \frac{dy(t)}{dt} \, dt$$

define inner products on $C^{(1)}[a,b]$, but

$$\int_a^b \frac{dx(t)}{dt}\,\frac{dy(t)}{dt}\,dt$$

is not an inner product on $C^{(1)}[a,b]$ because it contradicts axiom S4: from the $\langle x|x\rangle = 0$ it follows only $\frac{dx(t)}{dt} = 0$ and not $x(t) = 0$ everywhere in $[a,b]$.

Orthogonal vectors. The vectors x and y of an inner product space X are said to be orthogonal if

$$\langle x|y\rangle = 0\,.$$

Test for linear dependence. Using the inner product, an effective method can be given for determining whether or not the vectors $x_1,\ldots,x_n$ are linearly independent.

Theorem 8.1. Let X be an inner product space. A set of vectors $x_1,\ldots,x_n$ of X is linearly independent if and only if the *Gram matrix*

$$\begin{bmatrix} \langle x_1|x_1\rangle & . & . & . & \langle x_1|x_n\rangle \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ \langle x_n|x_1\rangle & . & . & . & \langle x_n|x_n\rangle \end{bmatrix}$$

is nonsingular.

Theorem 8.2. (*Cauchy-Schwarz inequality.*) For a real inner product space X,

$$\left|\langle x|y\rangle\right| \;\le\; \sqrt{\langle x|x\rangle}\,\sqrt{\langle y|y\rangle}\,, \qquad x, y \in X\,.$$

Proof: For any real scalar α,

$$0 \le \langle x-\alpha y|x-\alpha y\rangle = \langle x|x\rangle - \alpha\langle y|x\rangle - \alpha\langle x|y\rangle + \alpha^2\langle y|y\rangle,$$

that is

$$\langle y|y\rangle\alpha^2 - 2\langle x|y\rangle\alpha + \langle x|x\rangle \ge 0\,.$$

The left hand side of this inequality is a quadratic polynomial in variable α, having real coefficients. The inequality indicates that this polynomial cannot have *distinct* real zeros. Consequently, its discriminant cannot be positive, that is,

$$\left(2\langle x|y\rangle\right)^2 - 4\langle y|y\rangle\langle x|x\rangle \le 0,$$

from which the Cauchy-Schwarz inequality is immediate.

Note the general character[1] of this inequality and the simplicity of its proof.

Theorem 8.3. Every inner product space is a normed space. In fact, if x is an element in an inner product space X, the mapping

$$x \to \|x\| = \sqrt{\langle x|x\rangle}$$

defines a norm on X.

Proof: That $\sqrt{\langle x|x\rangle}$ satisfies the norm axioms N1 and N2, is obvious. The validity of the axiom N3 that is of the triangle inequality is easy to see using the Cauchy-Schwarz inequality as follows since:

$$\|x+y\|^2 = \langle x+y|x+y\rangle = \langle x|x\rangle + 2\langle x|y\rangle + \langle y|y\rangle \le \langle x|x\rangle + 2\left|\langle x|y\rangle\right| + \langle y|y\rangle$$

$$\le \langle x|x\rangle + 2\sqrt{\langle x|x\rangle}\sqrt{\langle y|y\rangle} + \langle y|y\rangle = \left(\|x\| + \|y\|\right)^2.$$

The existence of this norm makes possible to define the *completeness* of inner product spaces.

Hilbert space. A complete inner product space is called a *Hilbert space*.

Every finite-dimensional inner product space is complete; hence the *Euclidean space* is a Hilbert space.

We recall that the *Banach space* is a complete normed space. The Hilbert space is a special Banach space, where the norm $\|\cdot\|$ is induced by the inner product $\langle\cdot|\cdot\rangle$.

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[1] For example $\left|\sum_{i=1}^{n} x_i y_i\right| \le \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$ or $\left|\int_a^b x(t)\,y(t)\,dt\right| \le \sqrt{\int_a^b x^2(t)dt}\sqrt{\int_a^b y^2(t)dt}$

In the following we will use the fact that any continuous linear functional $F(u)$ can be written as a duality pair i.e. in the form $F(u)=\langle F \,|\, u\rangle$, $F \in X'$, $u \in X$, where X' is the dual space of the normed space X.

Theorem 8.4. (*Riesz*[1] *representation theorem*.) Let H be a Hilbert space, $F \in H'$. Then there is a unique $g \in H$ for which

$$F(u)=\langle g \,|\, u\rangle; \; u \in H .$$

In addition,

$$\|F\|_{H'} = \|g\|_{H} \; .$$

We want to emphasize that F is a continuous linear functional defined on the space H while g is an element of H.

If the linear functional is not bounded, then the Riesz theorem is not valid.

Proof: At first, assuming the existence of g, we prove its *uniqueness*. Suppose $\tilde{g} \in H$ satisfies

$$F(u)=\langle g \,|\, u\rangle = \langle \tilde{g} \,|\, u\rangle \quad \forall u \in H.$$

Then

$$\langle g-\tilde{g} \,|\, u\rangle = 0 \quad \forall u \in H .$$

Choose $u = g-\tilde{g}$. Then $\| g-\tilde{g} \| = 0$, which implies $g = \tilde{g}$.

We prove the *existence* of vector g.

Denote by

$$N = N(F) = \{u : u \in H, Fu = 0\},$$

the null space of F, which is a subspace of H. If $N = H$, then $Fu = 0$, and choosing $g = \Theta$ the proof is finished.

Now suppose $N \neq H$. Then there exists at least one $u_H \in H$ such that $F(u_H) \neq 0$. It is possible to decompose H as the direct sum $N \oplus M$, where $M = \{u \in H : \langle u \,|\, v\rangle = 0 \; \forall v \in N\}$ is the orthogonal complement of N with respect to H. Then we can write

[1] Frigyes Riesz (1880-1956), famous Hungarian mathematician.

$$u_H = u_N + u_M, \qquad u_N \in N, \qquad u_M \in M$$

where

$$F(u_M) = F(u_H - u_N) = F(u_H) - F(u_N) = F(u_H) \neq 0.$$

For any $u \in H$, it is true that

$$F\left(u - \frac{F(u)}{F(u_M)} u_M\right) = 0.$$

Hence $u - \frac{F(u)}{F(u_M)} u_M \in N$ and because u_M is orthogonal to N

$$\left\langle u - \frac{F(u)}{F(u_M)} u_M \,\middle|\, u_M \right\rangle = 0.$$

That is $\langle u | u_M \rangle - \frac{F(u)}{F(u_M)} \| u_M \|^2 = 0$ i.e.

$$F(u) = \left\langle \frac{F(u_M)}{\| u_M \|^2} u_M \,\middle|\, u \right\rangle.$$

In other words, we may choose g to be

$$\frac{F(u_M)}{\| u_M \|^2} u_M.$$

We complete the proof of the theorem by showing $\|F\|_{H'} = \|g\|_H$. From

$$F(u) = \langle g | u \rangle; \; u \in H$$

and the Cauchy-Schwarz inequality $|F(u)| = |\langle g | u \rangle| \leq \| g \|_H \| u \|_H, \quad u \in H$.

Hence (see the definition of norm of continuous linear functional)

$$\| F \|_{H^{1'}} = \sup_{u \neq \Theta} \frac{|\langle g | u \rangle|}{\| u \|_H} \leq \| g \|_H \quad .$$

However, it is impossible that $\| F \|_H < \| g \|_H$ because $F(g) = \langle g | g \rangle = \|g\|_H^2$ so that really $\| F \|_{H'} = \| g \|_H$ holds.

We note that this theorem is a fundamental tool in the solvability theory for elliptic partial differential equations.

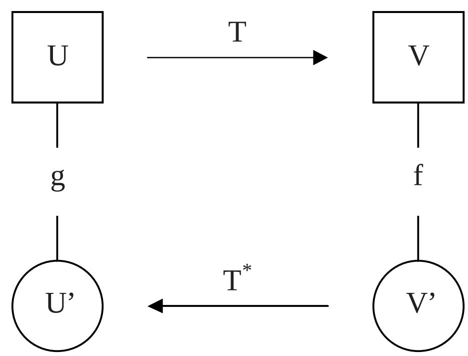
Adjoint operator. Consider a bounded (i.e. continuous) linear operator $T : U \to V$, where U and V are normed linear spaces. Let $f(v)$ be a continuous linear functional defined on V. Then $f \in V'$, where V' is the (topological) dual space of V. Since $v = Tu$, the $f(v)$ is defined for vectors $u \in U$, too. That is,

$$f\ (v) \equiv \langle f \,|\, v \rangle = \langle f \,|\, T\,u \rangle \equiv f\ (T\,u\ ) = g\ \ (u) \equiv \langle g \,|\, u \rangle$$

where $g(u)$ is a linear functional defined on U. The linear operator T is supposed to be bounded, hence using the Cauchy-Schwarz inequality we get

$$|\ g\ (u)| = \left| \langle\ f\ \mid Tu \rangle \right| \le \|\ f\ \| \, \|T\| \, \|u\| = K\ \|u\|, \text{ where } K = \|\ f\ \| \, \|T\|,$$

so that $g(u) = \langle g | u \rangle$ is bounded i.e. continuous linear functional on U. Hence $g \in U'$ where U' is the (topological) dual space of U.



Consequently, to each functional $f \in V'$ we have associated a corresponding functional $g \in U'$, that is, we have *defined* an operator which transforms the normed space V' to the normed space U'. This operator is denoted T^* and called the *adjoint* of T. That is, we can write $T^* f = g$.

We saw the equality $\langle f \,|\, T\,u \rangle = \langle g \,|\, u \rangle$ and substituting $T^* f = g$ we get

$$\langle\ f\ \mid Tu \rangle = \langle T^*\ f\ \mid u \rangle.$$

This equality also can be used as the definition of the adjoint operator.

If $U = V = H$ is Hilbert space, then the Riesz representation theorem guarantees the existence of a unique element $g = T^* f$ which satisfies the equation $\langle f \mid Tu \rangle = \langle g \mid u \rangle$.

The operator T is called *self-adjoint* if $T^* = T$.

E.8.3. For any matrix A, the *adjoint* of A is defined as the matrix A^*, which satisfies the equation

$$\langle Ax \mid y \rangle = \langle x \mid A^* y \rangle$$

for all $x, y \in C^n$, where $\langle u \mid v \rangle = \langle u^T \mid \bar{v} \rangle$ denotes the inner product of the complex euclidean space.

The matrix A is *self-adjoint* if $A^* = A$.

To find the adjoint matrix A^* explicitly, note that (using the Euclidean inner product)

$$\langle Ax \mid y \rangle = (Ax)^T \bar{y} = x^T A^T \bar{y} = x^T \overline{\left(\bar{A}^T y \right)} = \left\langle x \;\middle|\; \bar{A}^T y \right\rangle$$

so that $A^* = \overline{A}^T$, i.e. the adjoint of A is equal to the transpose of its conjugate.

If A is real symmetric, then $\overline{A}^T = A^T = A$, so that A is self-adjoint.

E.8.4. The *adjoint* of a differential operator L is defined to be the operator L^* for which

$$\langle Lu \mid v \rangle = \langle u \mid L^* v \rangle$$

for all u in the domain of L and v in domain of L^*.

This definition determines not only the operational definition of L^*, but its domain as well. Consider the example

$$Lu = \frac{du}{dx}$$

with the boundary condition $u(0) = 2u(1)$ with inner product

$$\langle u \mid v \rangle = \int_0^1 u(x) v(x) dx .$$

Using this inner product (and integration by parts) we have that

$$\langle Lu|v\rangle = \int_0^1 \frac{du}{dx}(x)v(x)dx = u(1)\,(v(1)-2v(0)) - \int_0^1 u(x)\frac{dv}{dx}(x)dx\,.$$

In order to make $\langle Lu|v\rangle = \langle u|L^*v\rangle$, we must take $L^*v = -\dfrac{dv}{dx}$ with the boundary condition $v(1) = 2\,v(0)$.

The operator L is *self-adjoint* if $L^* = L$. That means that not only the operational definitions of L and L^* agree but their domains are equal too.

An other application of the Riesz representation theorem is the *Lax-Milgram theorem* which is used traditionally to demonstrate existence and uniqueness of weak solutions of boundary value problems.

Theorem 8.5. (*Lax-Milgram theorem.*) Given an Hilbert space H with the inner product $\langle u|v\rangle$. Let $a\langle u|v\rangle$ be a bilinear functional in $u, v \in H$ such that

(1) $$\left|a\langle u|v\rangle\right| \le K\,\|u\|\,\|v\| \quad ,$$

(2) $$\left|a\langle u|u\rangle\right| \ge \alpha\,\|u\|^2 \quad ,$$

with positive constants K, α that are independent on u, v.

Further, let ℓ be a continuous linear functional on H, i.e. $\ell \in H'$. Then there *exists* a *unique* $u \in H$ such that

$$\ell(v) = a\langle u|v\rangle, \qquad v \in H\,.$$

9. Sets of measure zero, measurable functions

Set of measure zero. A set $A \subset [a,b]$ is said to be of measure zero, if for every $\varepsilon > 0$ there is a sequence of open intervals $I_k = (a_k, b_k)$ such as, that

$$\sum_{k=1}^{\infty} (b_k - a_k) \le \varepsilon \quad \text{and} \quad A \subseteq \bigcup_{k=1}^{\infty} I_k .$$

In other words, a set has measure zero if and only if it can be covered by a collection of open intervals whose total length is arbitrarily small.

Theorem 9.1. Any *countable* set $A = \{x_k;\ k = 1,2,...\} \subset [a,b]$ has a measure zero.

Proof: to any $\varepsilon > 0$ we can choose the intervals $I_k = (a_k, b_k)$ in the form

$$I_k = \left(x_k - \frac{\varepsilon}{2^{k+1}},\ x_k + \frac{\varepsilon}{2^{k+1}} \right), \qquad k = 1, 2, \ldots .$$

Then $\sum_{k=1}^{\infty} (b_k - a_k) = \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = \varepsilon \frac{1}{1 - \frac{1}{2}} \frac{1}{2} = \varepsilon$ and $A = \{ x_1, \ldots, x_k, \ldots \} \subseteq \bigcup_{k=1}^{\infty} I_k$ is valid too.

E.9.1. A finite set $A = \{ x_1, \ldots, x_n \} \subset [\, a,\, b\,]$ has measure zero.

The intervals $[a,b]$ and (a,b) have not measure zero. (Their common measure is their length: $b - a$.)

Almost everywhere property. A property $P = P(x)$ on an interval $[a,b]$ is said to hold *almost everywhere* on $[a,b]$, if P fails to hold only on subsets of $[a,b]$ of measure zero.

Measurable function. A (real-valued) function $f(x)$ defined on $[a,b]$ is said to be *measurable* on $[a,b]$, if there is a sequence of continuous functions $\{g_n(x) \colon a \le x \le b\}$, which (pointwise) converges to $f(x)$ almost everywhere on $[a,b]$.

In other terms, $f(x) = \lim_{n \to \infty} g_n(x)$ ($x \in [a,b] \setminus A$) where A is a set of measure zero. (The set A depends on $f(x)$.)

The set of measurable functions is a linear space.

We note that in the practice all functions are measurable.

The previous results for intervals can be extended to more-dimensional domains as well.

10. The space L_2

Let Ω denote a bounded domain (connected open set) in the n-dimensional Euclidean space.

Square integrable function. A measurable (real-valued) function $f(x), x \in \Omega$ is said to be *square integrable*[1], if

$$\int_{\Omega} f^2 dx < \infty .$$

The set $L_2(\Omega)$. The collection of all square integrable measurable (real-valued) functions defined on Ω is denoted $L_2(\Omega)$.

The $L_2(\Omega)$ is a very large set. It consists of all continuous functions on Ω and of all *bounded piecewise continuous* functions on Ω. The unbounded $f(x) = x^{-1/3}$ is also an element of $L_2(0,1)$ because $\int_0^1 \left(x^{-1/3}\right)^2 dx = 3$. However, the unbounded function $f(x) = x^{-1/2}$ does not belong to $L_2(0,1)$ since $\int_0^1 \left(x^{-1/2}\right)^2 dx = +\infty$.

We note that using *Mathematica* we got:

In[1]:= $\int_0^1 \left(\sqrt[-3]{x}\right)^2 dx$

Out[1]= 3

but

In[2]:= $\int_0^1 \left(x^{-\frac{1}{2}}\right)^2 dx$

— Integrate ::idiv : Integral of $\frac{1}{x}$ does not converge on {0, 1}. More…

Out[2]= $\int_0^1 \frac{1}{x} dx$

The sum of square integrable functions is a square integrable function as well. Indeed,

$$0 \le \left(|f| - |g|\right)^2 = f^2 - 2|f\,g| + g^2 \text{ and } f\,g \le |f\,g|$$

[1] in sense of Lebesgue integration , Henri Lebesgue (1875-1941)

so that

$$f\,g \le |\,f\,g| \le \frac{1}{2}\left(f^2 + g^2\right).$$

This inequality indicates, that

$$(f+g)^2 = f^2 + 2\,f\,g + g^2 \le 2\left(f^2 + g^2\right),$$

and after integration the assertion follows. Since the product of a square integrable function with a real number is a square integrable function, we have proved that $L_2(\Omega)$ is a *linear vector space*.

By integrating the last but one inequality it follows that the $\int_\Omega f\,g\,dx$ is *finite*, and hence there exists the *inner product*

$$\langle\, f \mid g\,\rangle = \int_\Omega f\,g\;dx\,, \quad f, g \in L_2(\Omega)\,.$$

In other words, the linear space of square integrable measurable functions $L_2(\Omega)$ is a *pre-Hilbert space*. But any inner-product space is a normed space, where the norm is defined using the inner product:

$$\|\,f\,\| = \sqrt{\langle\, f \mid f\,\rangle} = \left(\int_\Omega f^2\,dx\right)^{1/2}, \quad f \in L_2(\Omega).$$

If the integrals in this section are defined in the *Lebesgue sense*, then this real inner product space is also *complete*.

Theorem 10.1. The space $L_2(\Omega)$ with respect to the inner product above is a (real) *Hilbert space*.

In contrast with the linear space $C^{(0)}(\overline{\Omega})$ of continuous functions on a closed domain $\overline{\Omega}$, in the space $L_2(\Omega)$ the expressions such as the fourth axiom of the inner product or the first axiom of the norm need a more detailed commentary:

In $C^{(0)}(\overline{\Omega})$ the relation

$$\int_\Omega f^2\,dx = 0 \tag{10-1}$$

indicates $f \equiv 0$ on the closed domain $\overline{\Omega}$ and so on the open domain Ω too[1].

[1] Rememeber, that $C^{(0)}(\overline{\Omega}) \subset C^{(0)}(\Omega)$ because if $f \in C^{(0)}(\overline{\Omega})$ then $f \in C^{(0)}(\Omega)$.

In $L_2(\Omega)$, from (10-1) does not follow $f = 0$ everywhere in Ω.

Consider the function

$$f_1(x) = \begin{cases} 1, & \text{if} \quad x = \dfrac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

defined on $\Omega = (0,1)$. Obviously f_1 is square integrable on Ω and satisfies the equality (10-1). But this equality also satisfies, e.g. that function f_2, for which holds $f_2(x) = 0$ on $\Omega = (0,1)$ except on the *countable* set $\left\{ x_k = \dfrac{1}{k}; \quad k = 2, 3, \ldots \right\}$.

In general, the equality (10-1) satisfies all functions, for which $f(x) = 0$ on $\Omega = (0,1)$ except on a set of measure zero, that is $f(x) = 0$ *almost everywhere on* Ω. At the points where $f(x) \neq 0$ the values of f may be arbitrary, or f may be non-defined.

Equivalent functions. Two square integrable functions f and g are said to be *equivalent* if $f(x) = g(x)$ almost everywhere on Ω. Then we write also

$$f = g \qquad \text{in} \qquad L_2(\Omega)$$

and then obviously $\int_\Omega (f - g)^2 dx = 0$.

E.10.1. The *Dirichlet function*

$$f(x) = \begin{cases} 1 & \text{if} \quad x \quad \text{rational} \\ 0 & \text{if} \ x \ \text{irrational} \end{cases}; \qquad\qquad x \in [0,1]$$

is equal to zero *almost everywhere* on $[0,1]$ because the rational numbers are countable.

The *Riemann-integral* for this function does not exist, the *Lebesgue-integral* is equal to zero.

The proof of the following theorem is based on very serious results from the theory of measurable functions and the Lebesgue integral.

Theorem 10.2. Continuous functions are dense in $L_2(\Omega)$.

As already mentioned in example E 7.1, any continuous function on a finite interval $[a,b]$ is a limit of *uniformly* convergent and hence all the more in *mean* (in an integral norm) convergent sequence of polynomials. This fact and the theorem 10.2 imply:

Theorem 10.3. The polynomials are dense in $L_2(a,b)$.

Every polynomial is a linear combination of the power functions $1, t, t^2, \ldots,$ hence the sequence $\{t^k;\ k = 0, 1, 2, \ldots\}$ spans the space $L_2(a,b)$, and so - by theorem 7.3. - $L_2(a,b)$ is separable.

More generally:

Theorem 10.4. $L_2(\Omega)$ is separable.

11. Generalized derivatives, distributions, Sobolev spaces

Multi-index notation. The ordered n-tuple of non-negative integers

$$i=(i_1,i_2,\dots,i_n)$$

is called multi-index. The sum $i_1+i_2+\dots+i_n$ is denoted by $|\,i\,|$.

Using the multi-index notation, the partial derivatives of the function $f=f(x_1,x_2,\dots,x_n)$ can be expressed in shorter, so-called *operator* form

$$D^i f=\frac{\partial^{|i|} f}{\partial x_1^{i_1}\,\partial x_2^{i_2}\dots\partial x_n^{i_n}}.$$

E.11.1. If $i=(3,0)$, then instead of $D^i f=\dfrac{\partial^3 f}{\partial x_1^3\,\partial x_2^0}$ we write $D^i f=\dfrac{\partial^3 f}{\partial x_1^3}$.

If $|\,i\,|=0$, then $D^i f=f$.

If $n=2$, $k=3$ then $|\,i\,|\le k$ represents the following multi-indices $i=(i_1,i_2)$:

$(0,0)$	$\|\,i\,\|=0$
$(1,0),\ (0,1)$	$\|\,i\,\|=1$
$(2,0),\ (1,1),\ (0,2)$	$\|\,i\,\|=2$
$(3,0),\ (2,1),\ (1,2),\ (0,3)$	$\|\,i\,\|=3$

With this notation, for instance instead of

$$f(x_1,x_2)+\frac{\partial f}{\partial x_1}(x_1,x_2)+\frac{\partial f}{\partial x_2}(x_1,x_2)+\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2)+\frac{\partial^2 f}{\partial x_1\partial x_2}(x_1,x_2)+\frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)+$$
$$+\frac{\partial^3 f}{\partial x_1^3}(x_1,x_2)+\frac{\partial^3 f}{\partial x_1^2\partial x_2}(x_1,x_2)+\frac{\partial^3 f}{\partial x_1\,\partial x_2^2}(x_1,x_2)+\frac{\partial^3 f}{\partial x_2^3}(x_1,x_2)$$

we can use the short operator form $\sum\limits_{|\,i\,|\le k} D^i f$.

Smooth function. The function $\varphi:\Re^n\to\Re$ is said to be *smooth* if it is continuously differentiable infinitely times.

Support of function. Let Ω be a domain (i.e. a connected open subset) in $\Re^n$. The support of the function $\varphi(x)$, $x \in \Omega \subset \Re^n$, denoted by $\operatorname{supp}\varphi$ is the *closure* of the set of those points x at which $\varphi(x) \neq 0$. That is,

$$\operatorname{supp}\varphi = \overline{\{x : x \in \Omega,\ \varphi(x) \neq 0\}}.$$

Compact support. The function $\varphi(x)$, $x \in \Omega \subset \Re^n$ is said to have a *compact support* with respect to Ω, if its $\operatorname{supp}\varphi$ is a compact set (i.e., if it is bounded) and is a *proper* subset of Ω (i.e. if $\operatorname{supp}\varphi \subset \Omega$).

Since $\operatorname{supp}\varphi$ is a closed set by definition, and since it lies - by assumption - in an open domain Ω, it has a positive distance from the boundary $\partial\Omega$ of this domain.

Test function. Let Ω be a domain in $\Re^n$. Denote by $C_0^{(\infty)}(\Omega)$ the set of all smooth functions with compact support in Ω. That is, $C_0^{(\infty)}(\Omega) = \left\{\varphi \in C^{(\infty)}(\Omega) : \operatorname{supp}\varphi \subset \Omega\right\}$. Any $\varphi \in C_0^{(\infty)}(\Omega)$ is called *test function*.

E.11.2. In $\Re^1$ the

$$\varphi(x) = \begin{cases} e^{1/\left(x^2-1\right)} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

is an example of a smooth function with compact support in $\Omega = (-2,2)$. But $\varphi(x)$ is not a test function with respect to $\Omega = (-1,1)$, because the $\operatorname{supp}\varphi = \{-1 \leq x \leq 1\}$ is not a proper subset of the domain $\Omega = \{-1 < x < 1\}$.

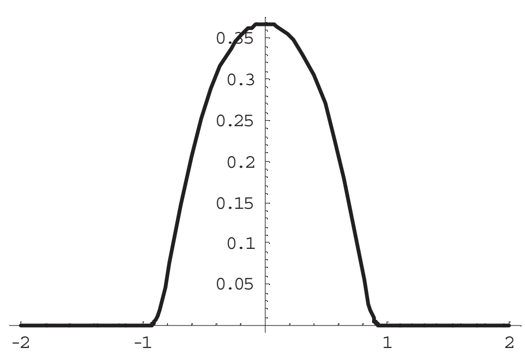
Using *Mathematica* we can easily plot this function:

In[1]:=

```
Clear[φ]
φ[x_] := Piecewise[{{0, x ≤ -1}, {E^(1/(x^2-1)), -1 < x < 1}, {0, x ≥ 1}}]
```

In[3]:=

```
Plot[φ[x], {x, -2, 2}, PlotStyle → {Thickness[0.008]}]
```



This function has limits at the points $x=-1,\ x=1$ *which are equal to zero. Indeed, e.g. at the point* $x=1$ *the limit from the left*

In[4]:= `Limit[φ[x], x→ 1, Direction→ 1]`

Out[4]= 0

and obviously the limit from the right

In[5]:= `Limit[φ[x], x→ 1, Direction→ -1]`

Out[5]= 0

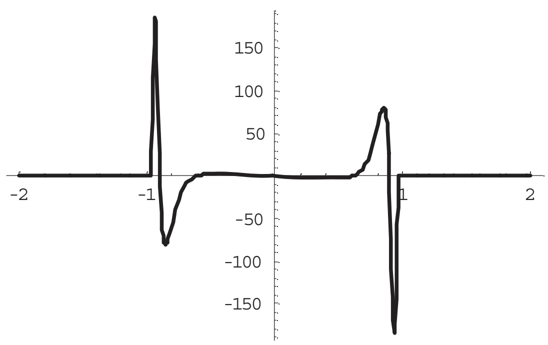
(The option *Direction* -1 and +1 denotes the right- and left-hand limit, respectively.) At the points $x=-1,\ x=1$ there exist all derivatives of $\varphi(x)$ and they are also equal to zero. Consequently $\varphi(x)$ is smooth too.

For example:

In[6]:= `D[φ[x], {x, 3}] // Simplify   (* The third derivative of φ *)`

Out[6]= $-\frac{4\, e^{\frac{1}{-1+x^2}}\, x\,(3-10x^2+3x^4+6x^6)}{(-1+x^2)^6} \qquad -1<x<1$

In[7]:= `Plot[%, {x, -2, 2}, PlotStyle→ {Thickness[0.008]}]`



Out[7]= - Graphics -

Suppose that the function $f=f(x_1,x_2,\ldots,x_n)$ in its domain Ω is *locally integrable*. That is, the *Lebesgue integral* of f in every compact subdomain of Ω is finite.

Generalized derivative. We say, that $D^i f$ is the i-th *generalized* or *distributional* or *weak* derivative of the function f in the domain Ω, if

$$\int_\Omega D^i f\ \varphi\ dx=(-1)^{|i|}\int_\Omega f\ D^i\varphi\, dx, \qquad (11\text{-}1)$$

where $\varphi \in C_0^{(\infty)}(\Omega)$, that is φ is an arbitrary test function.

It is easy to see, that if a function is differenciable, then its generalized derivative coincide with the classical one. For example, the integration by parts in $\Omega = (0,1)$ and using the test function property $\varphi(0) = \varphi(1) = 0$ leads to

$$\int_0^1 x^n \frac{d\varphi}{dx}\, dx = \left[x^n \varphi(x)\right]_0^1 - \int_0^1 n\, x^{n-1}\, \varphi(x)\, dx = (-1)\int_0^1 D^1 f\, \varphi\, dx\ .$$

E.11.3. Consider the function

$$f(x) = \begin{cases} c\,x & \text{for} \quad 0 < x \le \frac{1}{2} \\ c\,(1-x) & \text{for} \quad \frac{1}{2} \le x < 1 \end{cases}$$

where c is a constant. This function is not differentiable at $x = \frac{1}{2}$ (see the Figure 11.1).

The *piecewise* derivative of $f(x)$ on the interval $(0,1)$, coincides with its *generalized* derivative

$$D^{(1)} f = \begin{cases} c & \text{if} \quad 0 < x < \frac{1}{2} \\ -c & \text{if} \quad \frac{1}{2} < x < 1 \end{cases}.$$

.

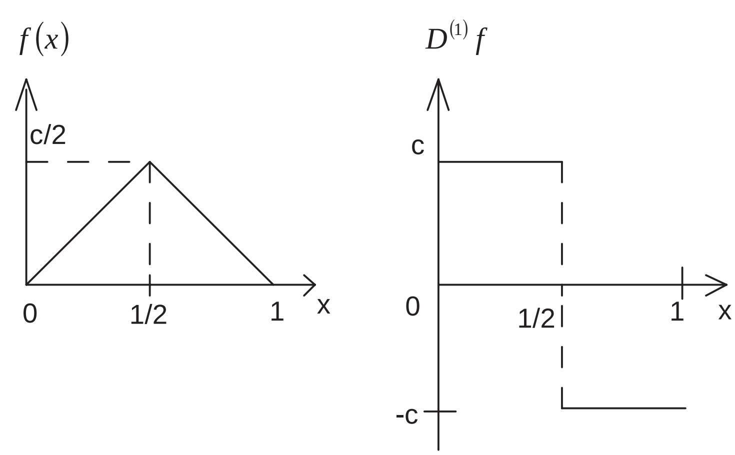


Figure 11.1. Non-differentiable functions at $x = \frac{1}{2}$.

Indeed, integration by parts yields to

$$\int_0^1 f\, D^{(1)}\varphi\, dx = \int_0^{1/2} c\, x \frac{d\varphi}{dx} dx + \int_{1/2}^1 c\, (1-x)\frac{d\varphi}{dx} dx =$$

$$= [c\, x\, \varphi(x)]_0^{1/2} - \int_0^{1/2} c\, \varphi(x) dx + [c(1-x)\varphi(x)]_{1/2}^1 - \int_{1/2}^1 (-c)\varphi(x)\, dx =$$

$$= \frac{c}{2}\varphi\left(\frac{1}{2}\right) - \int_0^{1/2} c\, \varphi(x)\, dx - \frac{c}{2}\varphi\left(\frac{1}{2}\right) - \int_{1/2}^1 (-c)\,\varphi(x)\, dx = (-1)\int_0^1 D^{(1)} f\, \varphi\, dx.$$

E.11.4. Now, consider the second generalized derivative $D^{(2)} f$ of the function $f(x)$ defined in $(0{,}1)$ in the previous example. The relation (11-1) - as definition- gives

$$\int_0^1 D^{(2)} f\, \varphi\, dx = (-1)^2 \int_0^1 f D^{(2)}\varphi\, dx = \int_0^{1/2} cx \frac{d^2\varphi}{dx^2}\, dx + \int_{1/2}^1 c(1-x)\frac{d^2\varphi}{dx^2}\, dx =$$

$$= \left[cx\frac{d\varphi}{dx}\right]_0^{1/2} - \int_0^{1/2} c\frac{d\varphi}{dx} dx + \left[c(1-x)\frac{d\varphi}{dx}\right]_{1/2}^1 - \int_{1/2}^1 (-c)\frac{d\varphi}{dx}\, dx =$$

$$= \frac{c}{2}\frac{d\varphi}{dx}\left(\frac{1}{2}\right) - [c\,\varphi(x)]_0^{1/2} - \frac{c}{2}\frac{d\varphi}{dx}\left(\frac{1}{2}\right) + c[\varphi(x)]_{1/2}^1 = -2c\,\varphi\left(\frac{1}{2}\right),$$

since $\varphi(x)$ is a test function on the interval $(0{,}1)$, that is, $\varphi(0) = \varphi(1) = 0$.

But no integrable function $D^{(2)} f$ exists which satisfies the definition (11-1), that is, the relationship

$$\int_0^1 D^{(2)} f\, \varphi(x)\, dx = -2c\,\varphi\left(\frac{1}{2}\right).$$

If we introduce the *symbolic notation*

$$\int_0^1 \delta\left(x - \frac{1}{2}\right) \varphi(x)\, dx = \varphi\left(\frac{1}{2}\right),$$

where the $\delta\left(x - \frac{1}{2}\right)$ is not a function but it is a so-called *delta-distribution*, then the second generalized derivative of the $f(x)$ given in the previous example is $D^{(2)} f = -2c\,\delta\left(x - \frac{1}{2}\right)$.

In applications it is often the case that the delta "function" is symbolically defined by

$$\delta(x-\xi)=0 \qquad \text{if} \qquad x\neq\xi \qquad \text{and} \qquad \int_{-\infty}^{\infty}\delta(x-\xi)\,dx=1.$$

However, for any reasonable definition of integration there cannot be such function. How is it, then, that many practicing scientists have used this definition with impunity for years?

To make mathematical sense of delta "function" there are, traditionally, two ways to do so, the first through delta sequences and the second through the theory of distributions.

The idea of *delta sequences* is to realize that, although a delta-function satisfying

$$\int_{-\infty}^{\infty}\delta(x-\xi)\;\varphi(x)\;dx=\varphi(\xi)$$

cannot exist, we might be able to find a sequence of functions $s_n(x-\xi)$ which in the limit $n\to\infty$ satisfies the defining equation

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}s_n(x-\xi)\varphi(x)\;dx=\varphi(\xi)$$

for all continuous functions $\varphi(x)$. Notice that this definition of s_n in no way implies that $\lim_{n\to\infty} s_n(x-\xi)=\delta(x-\xi)$ exists. In fact, we are certainly not allowed to interchange the limit pocess with integration. (We note that one criterion that allows the interchange of limit and integration is given by the Lebesgue theorem B.4. in Appendix B.)

There are many suggestive examples of delta sequences. The choice

$$s_n(x-\xi)=\begin{cases} n, \text{ if } & \xi-\dfrac{1}{2n}\leq x\leq\xi+\dfrac{1}{2n} \\ 0, \quad \text{if} & |x-\xi|>\dfrac{1}{2n}\end{cases}$$

(see Figure 11.2) is a delta sequence, since

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}s_n(x-\xi)\;\varphi(x)\;dx=\lim_{n\to\infty}\;n\int_{\xi-\frac{1}{2n}}^{\xi+\frac{1}{2n}}\varphi(x)\;dx=\varphi(\xi)$$

because of the mean value theorem[1], provided $\varphi(x)$ is continuous near $x = \xi$.

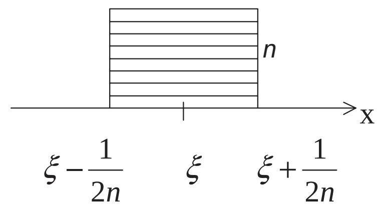


Figure 11.2. Delta sequence.

The application of delta sequences in practice is very cumbersome. For example, if we want to solve the boundary-value problem

$$\frac{d^2u}{dx^2} = \delta\left(x - \frac{1}{2}\right), \qquad 0 \le x \le 1$$

$$u(0) = u(1) = 0$$

(establish the shape of a cable under a concentrated force), then this problem can be approximated by the sequence of boundary-value problems

$$\frac{d^2u_n}{dx^2} = s_n\left(x - \frac{1}{2}\right), \qquad 0 \le x \le 1$$

$$u_n(0) = u_n(1) = 0.$$

Then, taking the limit

$$\lim_{n\to\infty} u_n(x) = u(x)$$

we can produce the correct solution for the examined boundary-value problem.

In more detail: If we solve the previous differential equation separately on the intervals

$$[\,0, \frac{1}{2} - \frac{1}{2n}), \quad [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}], \quad (\frac{1}{2} + \frac{1}{2n}, 1]$$

---

[1] there is such a point, $\eta \in [a,b]$, that $\int_a^b \varphi(x)\,dx = (b-a)\varphi(\eta)$

with respect to $u_n(0) = u_n(1) = 0$, then we have

$$u_n(x) = \begin{cases} ax, & \text{if } \ 0 \le x < \frac{1}{2} - \frac{1}{2n} \\ \frac{n}{2}x^2 + bx + c, & \text{if } \ \frac{1}{2} - \frac{1}{2n} \le x \le \frac{1}{2} + \frac{1}{2n} \\ d(x-1), & \text{if } \ \frac{1}{2} + \frac{1}{2n} < x \le 1 \end{cases}.$$

We can compute the coefficients a, b, c, d from the conditions for the continuity of values and the first derivatives of the function at $x = \frac{1}{2} - \frac{1}{2n}$ and $x = \frac{1}{2} + \frac{1}{2n}$.

The solution is the function

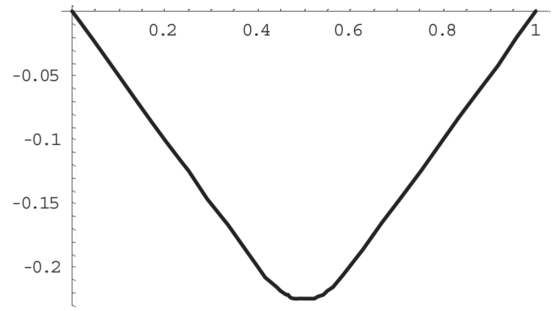
$$u_n(x) = \begin{cases} -\frac{1}{2}x, & \text{if } \ 0 \le x < \frac{1}{2} - \frac{1}{2n} \\ \frac{n}{2}\left[x(x-1) + \left(\frac{1}{2} - \frac{1}{2n} \right)^2 \right], & \text{if } \ \frac{1}{2} - \frac{1}{2n} \le x \le \frac{1}{2} + \frac{1}{2n} \\ \frac{1}{2}(x-1), & \text{if } \ \frac{1}{2} + \frac{1}{2n} < x \le 1 \end{cases}$$

which depends on n, is once continuously differenciable in the interval $[0,1]$ and satisfies the boundary conditions $u(0) = u(1) = 0$.

Using *Mathematica* it is easy to plot the function $u_n(x)$ and its first derivative according to x as follows:

```
In[1]:= u[n_, x_] := Piecewise[{{-x/2, 0 ≤ x < 1/2 - 1/(2 n)},
          {n/2 (x (x - 1) + (1/2 - 1/(2 n))^2), 1/2 - 1/(2 n) ≤ x ≤ 1/2 + 1/(2 n)},
          {(x - 1)/2, 1/2 + 1/(2 n) < x ≤ 1}}]
        v[x_] := u[5, x]
```

In[3]:= `Plot[v[x], {x, 0, 1}, PlotStyle → {Thickness[0.008]}]`

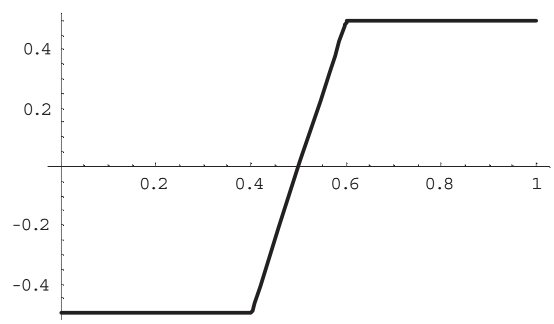


Out[3]= - Graphics -

In[4]:= `Derivative[1][v]`

Out[4]=
```
 0                 #1 < 0
 -1/2              0 < #1 < 2/5 || #1 == 2/5
 5/2 (-1 + 2 #1)   2/5 < #1 < 3/5
 1/2               #1 == 3/5 || 3/5 < #1 < 1            &
 0                 #1 > 1
 Indeterminate     True
```

In[5]:= `Plot[%[x], {x, 0, 1}, PlotStyle → {Thickness[0.008]}]`



Out[5]= - Graphics -

If $n \to \infty$, then $u_n \to u$ pointwise and hence in integral norm too, where

$$u = \begin{cases} -\frac{1}{2}x, & \text{if} \quad 0 \le x \le \frac{1}{2} \\ \frac{1}{2}(x-1), & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases}$$

(at the point $x = \frac{1}{2}$ we get $u_n\left(\frac{1}{2}\right) = -\frac{1}{4} + \frac{1}{8n}$ so that if $n \to \infty$, then $u_n\left(\frac{1}{2}\right) = -\frac{1}{4}$).

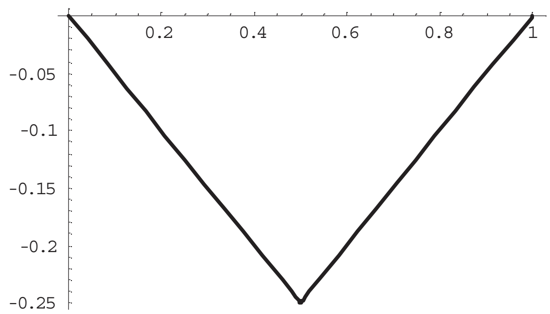
However, there is a further significant difficulty with this, since we do not only need to find $u_n(x)$, but also to show that the limit exists independently of the particular choice of delta sequence.

We note that using *Mathematica* this boundary-value problem can be solved easily as follows:

```
In[1]:= Clear[x, u]
        DSolve[{u''[x] == DiracDelta[x - 1/2], u[0] == 0, u[1] == 0}, u[x], x]

Out[2]= {{u[x] → 1/2 (-x - UnitStep[-1 + 2 x] + 2 x UnitStep[-1 + 2 x])}}

In[3]:= Plot[Evaluate[u[x] /. %], {x, 0, 1}, PlotStyle → {Thickness[0.008]}]
```




```
Out[3]= - Graphics -
```

A much more useful way to discuss delta "functions" is through the theory of distributions. As we shall see, distributions provide a generalization of functions and inner products.

Distributions.[1]

Convergence in the linear space $C_0^{(\infty)}(\Omega)$. In our previous discussions, the convergence was defined using the *norm*. In $C_0^{(\infty)}(\Omega) = \{\varphi \in C^{(\infty)}(\Omega) \colon \ \operatorname{supp} \varphi \subset \Omega\}$, that is in the set of all *smooth* functions with *compact support* in Ω, for our purposes suitable norm does not exist. Therefore, it is advantageous to introduce the following definition:

The sequence $\{\varphi_n\}$ of *test functions* is said to be *convergent* in $C_0^{(\infty)}(\Omega)$ and its limit is the test function φ, if there is a bounded set $\Omega^* \subset \Omega$, containing the supports of $\varphi, \varphi_1, \varphi_2, \ldots$ and if the sequence $\{\varphi_n\}$ and all its generalized derivatives

[1] The concept of distributions first was used by Laurent Schwartz in 1944.

$\{D^i\varphi_n\}$, $|i|>0$ converges uniformly to φ and its generalized derivatives $D^i\varphi$, respectively. That is, if

$$\lim_{n\to\infty}\max_{x\in\Omega^*}\left|D^i\varphi_n - D^i\varphi\right| = 0$$

for every i =0,1,2, … .

A *linear functional* $f: C_0^{(\infty)}(\Omega)\to\Re$ is said to be *continuous* (in $C_0^{(\infty)}(\Omega)$), if f maps every converegent sequence in $C_0^{(\infty)}(\Omega)$ into a convergent sequence in $\Re$, i.e. if $\langle f|\varphi_n\rangle \to \langle f|\varphi\rangle$ [1] whenever $\varphi_n\to\varphi$ in $C_0^{(\infty)}(\Omega)$.

Distribution. A continuous linear functional on $C_0^{(\infty)}(\Omega)$ is called *distribution* or *generalized function.*

E.11.5. An example of distribution is the delta-distribution δ, defined by

$$\langle\, \delta(x-\xi)\mid \varphi(x)\,\rangle = \varphi(\xi) \qquad \text{for all} \qquad \varphi\in C_0^{(\infty)}(\Omega).$$

The notation $\langle f|\varphi\rangle$ is used to denote the *"action"* of the distribution f on the test function φ.

The linearity of distributions means, that operations of addition and multiplication by scalar of distributions are defined as follows: if f and g are distributions (i.e. linear functionals on φ) and α and β are scalars, we define the distribution $\alpha f + \beta g$ to be the functional $\alpha\langle f|\varphi\rangle + \beta\langle g|\varphi\rangle$ for all $\varphi\in C_0^{(\infty)}(\Omega)$.

The notation of distributions looks exactly like an inner product between functions f and φ, and this similarity is intentional although misleading, since f need not be representable as a true inner product.

The simplest examples of linear functionals are indeed inner products. Suppose $f(x)$ is a *locally integrable*, that is, the Lebesgue integral $\int_I |f(x)|dx$ is defined and bounded for every finite interval I. Then the inner product

$$\langle f|\varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)dx$$

is a linear functional if φ is in $C_0^{(\infty)}(-\infty,\infty)$.

[1] The dual-pair notation $f(\varphi)\equiv\langle f|\varphi\rangle$ is used.

Every locally integrable function f induces a (so-called *regular*) distribution through the usual inner product. (Two locally integrable functions which are the same almost everywhere induce the same distribution.)

E.11.6. One important distribution is the *Heaviside-distribution*

$$\langle H \mid \varphi \rangle = \int_0^\infty \varphi(x)\,dx$$

which is equivalent to the inner product of φ with the well-known *Heaviside(Unitstep)- function*

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}.$$

For any function f that is locally integrable we can interchangeably refer to its function values $f(x)$ or to its distributional values (or action)

$$\langle f \mid \varphi \rangle = \int_{-\infty}^{\infty} f(x)\ \varphi(x)\,dx\,.$$

That is, either $x \to f(x)$, or $\varphi \to \langle f \mid \varphi \rangle$.

There are numerous distributions which are not representable as an inner product. The example we have already seen is the (so-called *singular*) distribution $\langle \delta_\xi \mid \varphi \rangle = \varphi(\xi)$. Similarly to the differential operator d/dx which cannot be evaluated at the point $x = 2$ for example, but d/dx can be evaluated pointwise only after it has first acted on a differentiable function $f(x)$, the distribution $\langle \delta_\xi \mid \varphi \rangle$ can be evaluated only after φ is known.

Sobolev spaces.

The set $C^{(k)}(\Omega)$, (as the linear space of k-times continuously differentiable functions on Ω), with respect to the inner product

$$\langle u \mid v \rangle_k = \sum_{|i| \leq k} \int_\Omega D^i u\, D^i v\, dx \qquad \text{(11-2)}$$

is a pre-Hilbert space with the norm

$$\| u \|_k = \sqrt{\langle u \mid u \rangle_k} = \left(\sum_{|i| \leq k} \int_\Omega \left(D^i u\right)^2 dx \right)^{1/2}$$

induced by (11-2).

Let $V_2^{(k)}(\Omega)$ denote this incomplete inner product space. The space $V_2^{(k)}(\Omega)$ can be completed by adding the limit points of all Cauchy sequences in $V_2^{(k)}(\Omega)$. These limit points are *distributions*.

The set D of all distributions generated by Cauchy sequences in $V_2^{(k)}(\Omega)$ is itself a linear space: if the distributions $f, g \in D$ are generated by $\{\varphi_n\}$ and $\{\psi_n\}$ respectively, then the distribution $\alpha f + \beta g$ is generated by the Cauchy sequence $\{\alpha\varphi_n + \beta\psi_n\}$. When the space D is endowed with the inner product

$$\langle f \mid g \rangle_k = \lim_{n\to\infty} \langle \varphi_n \mid \psi_n \rangle_k \tag{11-3}$$

where $\langle . \mid . \rangle_k$ is defined by (11-2), the resulting inner product space is the *Sobolev space* $H^{(k)}(\Omega)$.

It is in use to define the *Sobolev spaces* as follows too:

The *Sobolev space* $W_2^{(k)}(\Omega) = H^{(k)}(\Omega)$ is the set of those real-valued functions, which with their generalized derivatives up to and including the order k exist and are in Lebesgue sense square-integrable in Ω.

The space $W_2^{(k)}(\Omega)$ contains, besides the continuous functions, e.g. those functions, which are k-1-times continuously differentiable on $\overline{\Omega}$ and their derivative of order k is piecewise continuous in Ω.

The function f sketched in Figure 11.1. belongs to Sobolev space $W_2^{(1)}(0,1)$. Indeed, there exists its generalized derivative $D^{(1)}f$ which with the function f is square-integrable in $(0,1)$. The function f does not belong to $W_2^{(2)}(0,1)$, because its generalized derivative $D^{(2)}f$ exists only as distribution but is not a square-integrable function.

Theorem 11.1. The Sobolev space $W_2^{(k)}(\Omega) = H^{(k)}(\Omega)$ with the inner product (11-3) is a real Hilbert space.

12. Weak (or generalized) solutions

Consider the model boundary-value problem

$$-\frac{d^2u}{dx^2} = f, \qquad a \le x \le b \tag{12-1}$$

$$u(a) = u(b) = 0 .$$

If $f \in C^{(0)}(a,b)$, then the *classical solution* u (if exists) belongs to $u \in C^{(2)}(a,b) \cap C^{(0)}[a,b]$ and $u(a) = u(b) = 0$. Multiplying (12-1) by an arbitrary *test* function φ (recall, that $\varphi(x)$ is smooth with compact support in (a,b)) so that $\varphi(a) = \varphi(b) = 0$ and integrating the result, we obtain

$$-\int_a^b \frac{d^2u}{dx^2}\,\varphi\,dx = \int_a^b f\,\varphi\,dx. \tag{12-2}$$

Now an integration by parts yields

$$\int_a^b \frac{du}{dx}\frac{d\varphi}{dx}\,dx = \int_a^b f\,\varphi\,dx, \qquad \varphi \in C_0^{(\infty)}(a,b). \tag{12-3}$$

If $f \notin C^{(0)}(a,b)$ then the equation (12-1) does not have a classical solution (because then $u \notin C^{(2)}(a,b)$). For such a case it is possible to weaken, generalize the concept of the solution. Note, that for $f \in L_2(a,b)$ the equation (12-3) makes sense if $\frac{du}{dx} \in L_2(a,b)$. This requirement is satisfied if

$$u \in \overset{0}{W}{}_2^{(1)}(a,b) = \left\{u\!: \ u \in W_2^{(1)}(a,b), \ u(a) = u(b) = 0\right\}$$

where the symbol $\overset{0}{\ }$ refers to homogeneous boundary conditions. Then the definition of *Sobolev space* $W_2^{(1)}(a,b)$ implies that the derivatives are considered in the generalized sense and that $\frac{du}{dx} \in L_2(a,b)$.

The function $u \in \overset{0}{W}{}_2^{(1)}(a,b)$ is called the *weak* or *generalized solution* of equation (12-1), if for $f \in L_2(a,b)$, u satisfies equation (12-3). In other terms, a generalized solution of equation (12-1) is a distribution $u \in \overset{0}{W}{}_2^{(1)}(a,b)$ such that equation (12-2) or equivalently, equation (12-3), is satisfied for all $\varphi \in C_0^{(\infty)}(a,b)$ and a given distribution f in $L_2(a,b)$.

We note that $C_0^{(\infty)}(a,b)$ is a *dense* subspace of $\overset{0}{W}{}_2^{(1)}(a,b)$ (because $\overset{0}{W}{}_2^{(1)}(a,b)$ is the closure (extension) of $C_0^{(\infty)}(a,b)$). This property guaranties that if u is a classical solution of the equation (12-1) then it is also a solution of the weak formulation (12-3). Indeed, the integration by parts on the left-hand side of (12-3) yields

$$\int_a^b \left(\frac{d^2u}{dx^2} + f \right) \varphi \; dx = 0 \quad \varphi \in C_0^{(\infty)}(a,b).$$

By the lemma[1] known from the calculus of variations we must have

$$\frac{d^2u}{d\,x^2} + f = \Theta \qquad \text{in} \qquad L_2(a,b)$$

and the continuity of f implies that the differential equation (12-1) is satisfied.

The equation (12-3) can be expressed in the form

$$\left\langle D^{(1)}(u) \middle| D^{(1)}(\varphi) \right\rangle = \langle f \mid \varphi \rangle, \qquad \varphi \in C_0^{(\infty)}(a,b)$$

where $\langle \cdot | \cdot \rangle$ is the inner product in the space $L_2(a,b)$. This equation makes sense when φ is any element of $\overset{0}{W}{}_2^{(1)}(a,b)$. Since $C_0^{(\infty)}(a,b)$ is a dense subspace of $\overset{0}{W}{}_2^{(1)}(a,b)$, it follows that this equation is equivalent to

$$\left\langle D^{(1)}(u) \middle| D^{(1)}(\varphi) \right\rangle = \langle f \mid \varphi \rangle, \qquad \varphi \in \overset{0}{W}{}_2^{(1)}(a,b).$$

Denote $H = \overset{0}{W}{}_2^{(1)}(a,b)$, and let $a\langle . | . \rangle : H \times H \to \Re$ be the bilinear functional defined by

$$a\langle u | v \rangle = \left\langle D^{(1)}(u) \middle| D^{(1)}(v) \right\rangle \quad \text{for} \quad u, v \in H,$$

and let $\ell : H \to \Re$ be the continuous linear functional on H (i.e. $\ell \in H'$) defined by

$$\ell(v) = \langle f | v \rangle \quad \text{for} \quad \forall v \in H \,.$$

[1] Lemma: If $u \in H$ is orthogonal to all elements v of a set S which is dense in a pre-Hilbert space H, then u is the null-element of H.

Then the *weak formulation* of the boundary-value problem is to find $u \in H$ such that

$$a\langle u \,|\, v\rangle = \ell(v) \qquad \forall v \in H \ . \tag{12-4}$$

The solvability of the problem (12-4) is given in the *Lax-Milgram theorem* (see theorem 8.5) which is a generalization of the Riesz representation theorem.

Weak convergence. A sequence of distributions $\{f_n\}$ is said to converge to the distribution f if their actions converge in $\Re$, that is, if

$$\langle f_n \,|\, \varphi \rangle \to \langle f \,|\, \varphi \rangle \quad \text{for all} \qquad \varphi \in C_0^{(\infty)}(\Omega).$$

This convergence is called *convergence in the sense of distribution* or *weak convergence.*

If the sequence of distributions f_n converges to f then the sequence of derivatives $D^{(1)} f_n$ converges to $D^{(1)} f$. This follows since

$$\langle D^{(1)} f_n \,|\, \varphi \rangle = -\langle f_n \,|\, D^{(1)}\varphi \rangle \to -\langle f \,|\, D^{(1)}\varphi \rangle = \langle D^{(1)} f \,|\, \varphi \rangle$$

for all $\varphi \in C_0^{(\infty)}$.

E.12.1. The sequence $\{f_n\} = \left\{\dfrac{\cos nx}{n}\right\}$ is both a sequence of functions and a sequence of distributions. As $n \to \infty$, f_n converges to 0 both as a function (pointwise) and as a distribution. It follows that $D^{(1)} f_n = -\sin nx$ converges to the zero distribution even though the pointwise limit is not defined.

Recall, that using distributions, we are able to generalize the concept of function and derivative to many objects which previously made no sense in the usual definitions. It is also possible to generalize the concept of a differential equation.

Weak formulation. The differential equation $Lu = f$ is a differential equation in the sense of distribution (i.e., in the weak sense), if f and u are distributions and all derivatives are interpreted in the sense of distributions. Such a differential equation is called the *weak formulation* of the differential equation.

13. Orthogonal systems, Fourier series

Orthonormal systems. A sequence $\{\varphi_k;\ k=1,2,\ldots\}$ of elements of an *inner product* space is said to be *orthonormal*, if

$$\langle \varphi_i \mid \varphi_j \rangle = \begin{cases} 0 & \text{if } \ i \neq j \\ 1 & \text{if } \ i = j \end{cases}.$$

The vectors of an orthonormal system are *linearly independent*. Indeed, if we select n vectors from an orthonormal system $\{\varphi_k;\ k=1,2,\ldots\}$, then they are linearly independent. This follows from the Theorem 8.1 because the corresponding Gram matrix is equal to the identity matrix.

Fourier series. Suppose, that $\{\varphi_k;\ k=1,2,\ldots\}$ is an orthonormal system in a Hilbert space H and that u is an arbitrary element of H. The series

$$\sum_{k=1}^{\infty} \alpha_k \varphi_k; \qquad \alpha_k = \langle u \mid \varphi_k \rangle \tag{13-1}$$

is said to be an *orthogonal* or *generalized Fourier series representation* of $u \in H$, with respect to the orthonormal system $\{\varphi_k;\ k=1,2,\ldots\}$, and the scalars $\alpha_k;\ k=1,2,\ \ldots$ are called the (generalized) *Fourier coefficients* of $u \in H$.

Convergence of infinite series. Let $\{u_k;\ k=1,2,\ldots\}$ denote a sequence of vectors in a normed space U. An infinite series

$$\sum_{k=1}^{\infty} u_k$$

is said to be *convergent* if and only if the sequence of n-th partial sums

$$s_n = \sum_{k=1}^{n} u_k$$

converges. In other words, an infinite series $\sum_{k=1}^{\infty} u_k$ converges, if and only if there exists a vector $s \in U$ such, that for every $\varepsilon > 0$ there is an integer $N > 0$ such, that

$$\| s_n - s \| < \varepsilon \qquad \text{whenever} \quad n \geq N .$$

Theorem 13.1. Let $\{\varphi_k;\ k=1,2,\ldots\}$ be an orthonormal system in a Hilbert space H. Then the Fourier series representation (13-1) of an arbitrary element $u\in H$ converges.

However, it does not follow, that the limit of this series is u !

E.13.1. The sequence

$$\left\{\ \varphi_k=\sqrt{\frac{2}{\pi}}\ \sin\left(\left(2k-1\right)x\right);\quad k=1,2,\ldots\right\} \tag{13-2}$$

is orthonormal in the Hilbert space $H=L_2(0,\pi)$. Indeed, using *Mathematica* we have got

```
In[1]:=  φk_ := Sqrt[2/π] Sin[(2 k - 1) x]
         Assuming[{m, n} ∈ Integers, {∫_0^π φm φn dx, ∫_0^π φn^2 dx}]
Out[2]=  {0, 1}
```

The function

$$u=\sin 2x$$

is orthogonal to every φ_k and hence the Fourier coefficients of u

$$\alpha_k=\langle u\,|\,\varphi_k\rangle=\int_0^\pi \sin 2x\,\varphi_k\,dx=2\int_0^\pi \varphi_k\sin x\;d(\sin x)=2\int_0^0 \varphi_k\,t\;dt=0\,.$$

The corresponding Fourier series

$$0\sqrt{\frac{2}{\pi}}\sin x+0\sqrt{\frac{2}{\pi}}\sin 3x+0\sqrt{\frac{2}{\pi}}\sin 5x+\ldots$$

converges - in accordance with the Theorem 13.1 - but its limit is the function $u=\Theta$ and not $u=\sin 2x$.

This does not occure, if the sequence (13-2) would contain the function $\sqrt{\frac{2}{\pi}}\sin 2x$ too. Hence the sequence (13-2) is in this respect deficient, *"incomplete"*.

Complete *orthonormal* sets. Let $\{\varphi_k;\ k=1,2,\ldots\}$ be an *orthonormal* set in a Hilbert space H. The orthonormal set $\{\varphi_k\}$ is said to be *complete*, if for every element $u \in H$ the Fourier series representation of u with respect to $\{\varphi_k;\ k=1,2,\ldots\}$ converges to the limit u.

Do not be confused by this second use of the notion *complete*. In the Section 5 we defined, that a (normed) *space* S is complete if every Cauchy sequence in S is convergent in S. Here we say that the *set* $\{\varphi_k;\ k=1,2,\ldots\}$ is complete, if $\sum_{k=1}^{\infty}\langle u|\varphi_k\rangle\varphi_k = u$ for every u in the Hilbert space H.

Theorem 13.2. An orthonormal system $\{\varphi_k;\ k=1,2,\ldots\}$ is *complete* if and only if

$$\langle u|\varphi_k\rangle = 0 \text{ for all } k \quad \text{implies} \quad u=\Theta.$$

That is, the orthonormal system $\{\varphi_k;\ k=1,2,\ldots\}$ is *complete*, if $u=\Theta$ is the only vector in H which is orthogonal to all elements of the set $\{\varphi_k;\ k=1,2,\ldots\}$.

We saw that the function $u=\sin 2x \neq \Theta$ is orthogonal to all elements of the orthonormal set (13-2). Hence also by theorem 13.2 the set (13-2) is not complete.

Recall, that a subset S of a Hilbert space H *spans* or *generates* the Hilbert space, if the set of all linear combinations of the elements of S is *dense* in H.

Complete sequences. Let H be a Hilbert space. A sequence

$$\{\psi_k;\quad k=1,2,\ldots\} \tag{13-3}$$

of (not only mutually orthogonal) elements of H is called a *complete* sequence of H, if $\{\psi_k;\quad k=1,2,\ldots\}$ *spans* the Hilbert space H.

In other words, $\{\psi_k;\quad k=1,2,\ldots\}$ is a *complete sequence* of a Hilbert space H, if for any $u \in H$ and for every $\varepsilon > 0$, there is a positive integer N and numbers $a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)}$ such, that

$$\left\| u - \sum_{k=1}^{n} a_k^{(n)}\psi_k \right\| < \varepsilon \text{ for all } n \geq N.$$

Bases for a Hilbert space. Let H be a Hilbert space. A sequence $\{\psi_k;\quad k=1,2,\ldots\}$ of elements of H is called a *basis* for H, if $\{\psi_k;\quad k=1,2,\ldots\}$

1. is a *complete sequence* of H

(that is, every $u \in H$ can be approximated with an arbitrary accuracy

by linear combinations of elements of $\{\psi_k;\quad k=1,2,\ldots\}$)

and

2. is *linearly independent*.

In other tems, a basis for a Hilbert space H is every linearly independent countable subset of H which spans H.

We note, that in the finite element approximations not only the coefficients $a_1^{(n)}, a_2^{(n)}, \dots, a_n^{(n)}$ but also the basis functions $\psi_1^{(n)}, \psi_2^{(n)}, \dots, \psi_n^{(n)}$ depend on the accuracy ε.

Schauder basis. If the basis for a Hilbert space H has the property, that any $u \in H$ can be *uniquely* written in the form of infinite sum

$$u = \sum_{k=1}^{\infty} a_k \psi_k$$

(which is interpreted as $\left\| u - \sum_{k=1}^{n} a_k \psi_k \right\| < \varepsilon$ for all $n > N$) then the linearly independent sequence $\{\psi_k; \quad k = 1, 2, \dots\}$ is called *Schauder basis.*

If a sequence (13-3) is *orthonormal* then the previous definitions of *complete sequences* and of *complete orthonormal sets* express *identical* concepts.

Orthonormal bases. An orthonormal sequence

$$\{\varphi_k;\ k = 1, 2, \dots\}$$

that forms a basis for a Hilbert space H is called an orthonormal basis for H.

Now we give an algorithm for the construction of an orthonormal basis $\{\varphi_k;\ k = 1, 2, \dots\}$ once a basis $\{\psi_k; \quad k = 1, 2, \dots\}$ for H is given.

Gram-Schmidt orthonormalization. Let

$$\varphi_1 = \frac{\psi_1}{\|\psi_1\|}$$

be the first element of the orthonormal set. (The linear independence of $\{\psi_k; \quad k = 1, 2, \dots\}$ implies that the zero vector is not an element of $\{\psi_k\}$). Obviously $\| \varphi_1 \| = 1$.

Let

$$g_2 = \psi_2 + c_{12} \varphi_1$$

be orthogonal to φ_1 (see Fig. 13.1), that is let

$$\langle \varphi_1 \mid g_2 \rangle = \langle \varphi_1 \mid \psi_2 + c_{12}\varphi_1 \rangle = 0,$$

from which it follows $c_{12} = -\langle \varphi_1 \mid \psi_2 \rangle$.

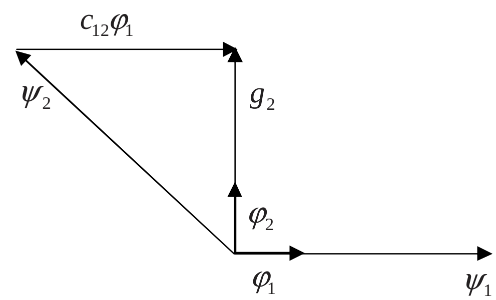


Figure 13.1. First step of the orthonormalization procedure

We note, that $\| g_2 \| \neq 0$ because if

$$g_2 = \psi_2 + \frac{c_{12}}{\| \psi_1 \|} \psi_1 = \Theta$$

then the vectors ψ_1, ψ_2 are not linearly independent. Hence

$$\varphi_2 = \frac{g_2}{\| g_2 \|}$$

and φ_1 and φ_2 are mutually orthogonal vectors and have unit norms.

Let

$$g_3 = \psi_3 + c_{13}\varphi_1 + c_{23}\varphi_2$$

be orthogonal to φ_1 and to φ_2, that is let

$$\langle \varphi_1 \mid g_3 \rangle = \langle \varphi_1 \mid \psi_3 \rangle + c_{13} \langle \varphi_1 \mid \varphi_1 \rangle + c_{23} \langle \varphi_1 \mid \varphi_2 \rangle = 0$$

$$\langle \varphi_2 \mid g_3 \rangle = \langle \varphi_2 \mid \psi_3 \rangle + c_{13} \langle \varphi_2 \mid \varphi_1 \rangle + c_{23} \langle \varphi_2 \mid \varphi_2 \rangle = 0$$

from which it follows

$$c_{13} = -\langle \varphi_1 \mid \psi_3 \rangle, \; c_{23} = -\langle \varphi_2 \mid \psi_3 \rangle .$$

Similarly to the case of the vector g_2 it can be shown, that $g_3 \neq \Theta$ and hence

$$\varphi_3 = \frac{g_3}{\| g_3 \|},$$

φ_1, φ_2 and φ_3 are mutually orthogonal and have unit norms.

If we continue this technique, then we get an orthonormal system $\{\varphi_1, \varphi_2, ...\}$.

It is easy to see, that the sequence $\{\varphi_k;\; k = 1, 2, ...\}$ is an *orthonormal basis* for a Hilbert space H. The orthonormality of the sequence immediately follows from its construction, so that it is enough to prove that it is *complete*. From the process of the orthonormalization it is clearly seen, that the n-th element of $\{\varphi_k;\; k = 1, 2, ...\}$ is a linear combination of the first n elements of $\{\psi_k;\; k = 1, 2, ...\}$ and vice-versa. Therefore, if any $u \in H$ can be approximated with an arbitrary accuracy by linear combinations of the elements of $\{\psi_k;\; k = 1, 2, ...\} \subset H$ then this is also true for the linear combinations of the elements of $\{\varphi_k;\; k = 1, 2, ...\} \subset H$, that means that the sequence $\{\varphi_k;\; k = 1, 2, ...\}$ is complete in H.

E.13.2. Recall, that the sequence of functions

$$\{1,\; x,\; x^2, x^3, x^4, ...\}$$

forms a base for the Hilbert space $L_2(-1,1)$. But this base is not orthonormal. To orthonormalize it we may apply the process of Gram-Schmidt orthonormalization which can be realized using *Mathematica* as follows:

```
In[1]:= << LinearAlgebra`Orthogonalization`
        {y1, y2, y3, y4, y4} =
         GramSchmidt[{1, x, x^2, x^3, x^4}, Normalized → True,
          InnerProduct → (∫_{-1}^{1} (#1 #2) dx &)]
```

Out[2]=
$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \frac{3}{2}\sqrt{\frac{5}{2}}\left(-\frac{1}{3}+x^2\right), \frac{5}{2}\sqrt{\frac{7}{2}}\left(-\frac{3x}{5}+x^3\right), \frac{105\left(-\frac{1}{5}+x^4-\frac{6}{7}\left(-\frac{1}{3}+x^2\right)\right)}{8\sqrt{2}}\right\}$$

Control:

In[3]:= $\left\{\int_{-1}^{1} y1^2\, dx,\ \int_{-1}^{1} y1\, y2\, dx\right\}$

Out[3]= {1, 0}

The following example applies the Gram-Schmidt orthonormalization to the given list of three-dimensional vectors.

```
In[1]:= << LinearAlgebra`Orthogonalization`
        u1 = {3, 4, 2}; u2 = {2, 5, 2}; u3 = {1, 2, 6};
        {v1, v2, v3} = GramSchmidt[{u1, u2, u3}]
```

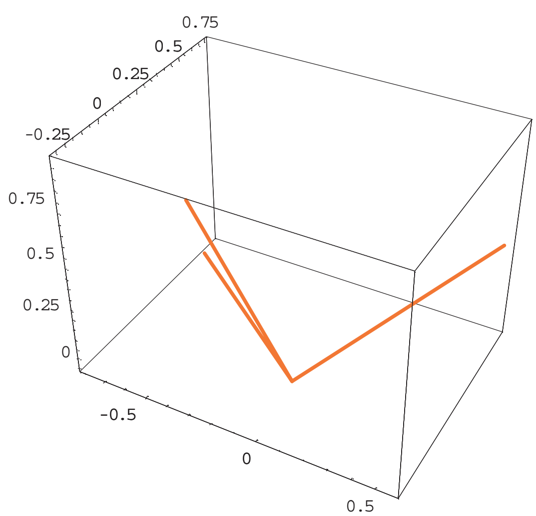
Out[3]= $\left\{\left\{\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right\}, \left\{-\frac{32}{\sqrt{1653}}, \frac{25}{\sqrt{1653}}, -\frac{2}{\sqrt{1653}}\right\}, \left\{-\frac{2}{\sqrt{57}}, -\frac{2}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right\}\right\}$

The result is an orthonormal basis, so the scalar product of each pair of vectors is zero and each vector has unit length:

```
In[4]:= {v1.v2, v2.v3, v1.v3, v1.v1, v2.v2, v3.v3}
```

Out[4]= {0, 0, 0, 1, 1, 1}

```
In[5]:= Show[
         Graphics3D[
          {Hue[0], Thickness[0.008], Map[Line[{{0, 0, 0}, #}] &, {v1, v2, v3}]}],
         Axes → True]
```



Out[5]= - Graphics3D -

Theorem 13.3. If a Hilbert space H is *separable*, then H contains a *basis*.

Proof: If a Hilbert space is separable, then H contains a countable *dense* subset which spans H. If we order the elements of this subset into a sequence, then we have a *complete sequence;* if we omit those elements which are linear combinations of the others, then this does not disturb the completeness of the sequence, so that we have a linearly independent complete sequence of H, which is a basis for H.

Theorem 13.3 and the Gram-Schmidt orthonormalization implies:

Theorem 13.4. A separable Hilbert space has an orthonormal basis.

14. The projection theorem, the best approximation

From simple geometry it is well known how to determine the point m^* among all poits m of a plane M, for which the distance from a point $x \notin M$ is the shortest. The point m^* is given as the intersection of a line passing through the point x, perpendicular to the plane M itself (see Fig. 14.1).

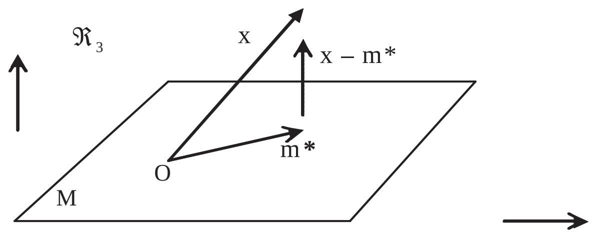


Figure 14.1. Orthogonal projection in $\Re_3$.

This obvious and intuitive result can be generalized to the problem of finding *the best approximation* m^* of a given vector x of a Hilbert space in the subspace M.

Minimizing vector. Let H be an inner product space and M a linear subspace of H. A vector $m^* \in M$ is said to be a minimizing vector of a given vector $x \in H$ in a subspace M, if

$$\| x - m^* \| \leq \| x - m \| \qquad \text{for all} \quad m \in M .$$

Theorem 14.1. (The projection theorem).

(1) $m^* \in M$ is a minimizig vector of a vector $x \in H$ in a subspace M, if and only if

$$\langle x - m^* \mid m \rangle = 0 \qquad \text{for all} \quad m \in M .$$

(2) If a minimizing vector exists, then it is *unique*.

(3) If H is a Hilbert space, and M is a *closed* linear subspace in H (i.e. if $m_n \in M$ and $m_n \to m$, implies $m \in M$), then to every $x \in H$ there *exists* a minimizing vector m^* in M.

We note, that neither the orthogonality of $x-m^*$ nor the unicity of m^* depends on the completeness of the space H.

The best approximation. Let H be a real Hilbert space and suppose we have an orthonormal sequence of functions $\{\varphi_k;\ k=1,2,...,n\}\subset H$ with which we wish to approximate an arbitrary function u in H. To approximate u in the best possible way, we want a linear combination of $\{\varphi_k;\ k=1,2,...,n\}$ which is as close as possible, in terms of the norm in H, to u. In other words, we want to minimize

$$\left\| u-\sum_{k=1}^{n} c_k\varphi_k \right\|.$$

By the projection theorem, this norm is minimal if and only if

$$\left\langle u-\sum_{k=1}^{n} c_k\,\varphi_k \middle| \varphi_i \right\rangle = 0,\quad i=1,2,...,n$$

Since the orthonormality of the sequence $\{\varphi_k;\ k=1,2,...,n\}$, it follows that $c_k=\langle u|\varphi_k\rangle$, $k=1,2,\ ...\ ,n$, that is, the scalars $c_k;\ k=1,2,\ ...\ ,n$ are the Fourier coefficients of $u\in H$.

With this choice of c_k, the error of our approximation is

$$\left\| u-\sum_{k=1}^{n} c_k\varphi_k \right\|^2 = \left\langle u-\sum_{k=1}^{n} c_k\varphi_k \middle| u-\sum_{k=1}^{n} c_k\varphi_k \right\rangle = \|u\|^2-\sum_{k=1}^{n}\langle u|\varphi_k\rangle^2 .$$

Since the error (norm) can never be negative, it follows that

$$\sum_{k=1}^{n} c_k^2=\sum_{k=1}^{n}\langle u|\varphi_k\rangle^2\le\|u\|^2<\infty,$$

which is known as *Bessel's inequality*. Since this is true for all n, if the set $\{\varphi_k;\ k=1,2,...,n\}$ is infinite, we can take the limit $n\to\infty$, and conclude that $\sum_{k=1}^{\infty} c_k\varphi_k$ converges to some function in H. (We know this because $\sum_{k=1}^{n} c_k\varphi_k$ is a Cauchy sequence in H and H is complete.)

For any orthonormal system $\{\varphi_k;\ k=1,2,...,n\}$, the best approximation of u is $\sum_{k=1}^{n}\langle u|\varphi_k\rangle\varphi_k$, which is the projection of u onto the space spanned by the set $\{\varphi_k;\ k=1,2,...,n\}$

Appendix: A. Construction of basis functions for the vector space $C_3^{(1)}[-1,1]$ using Mathematica

Training problem: Let $C_3^{(1)}[-1,1]$ denote that linear subspace of the vector space $C^{(1)}[-1,1]$, which consists of piecewise cubic polynomials on the intervals $[-1,0]$ and $[0,1]$. Similarly to example E.2.2, determine the basis of the function space $C_3^{(1)}[-1,1]$ and plot it using *Mathematica*!

```
In[1]:= Clear["Global`*"]     (* Training problem *)

        p1[t_] := a1 t^3 + b1 t^2 + c1 t + d1
        p2[t_] := a2 t^3 + b2 t^2 + c2 t + d2

        (* conditions for continuity in t==0 *)

        p1[0] == p2[0]

Out[4]= d1 == d2

In[5]:= (D[p1[t], t] /. t → 0) == (D[p2[t], t] /. t → 0)

Out[5]= c1 == c2

In[6]:= d1 = d2 = d; c1 = c2 = c;

        p[t_] := Piecewise[{{p1[t], -1 ≤ t ≤ 0}, {p2[t], 0 ≤ t ≤ 1}}]

        dp[t_] := Piecewise[{{p1'[t], -1 ≤ t ≤ 0}, {p2'[t], 0 ≤ t ≤ 1}}]

        s = Solve[{Map[p, {-1, 0, 1}] == Map[g, {-1, 0, 1}],
             Map[dp, {-1, 0, 1}] == Map[dg, {-1, 0, 1}]} // Flatten, {a1, b1, a2, b2, c, d}] //
          First

Out[9]= {a1 → dg[-1] + dg[0] + 2 g[-1] - 2 g[0],
         b1 → dg[-1] + 2 dg[0] + 3 g[-1] - 3 g[0], a2 → dg[0] + dg[1] + 2 g[0] - 2 g[1],
         b2 → -2 dg[0] - dg[1] - 3 g[0] + 3 g[1], c → dg[0], d → g[0]}

In[10]:= (* substitution of the solution into p[t_] *)

         ss = p[t] /. s

Out[10]= { t dg[0] + t^2 (dg[-1] + 2 dg[0] + 3 g[-1] - 3 g[0]) + t^3 (dg[-1] + dg[0] + 2 g[-1] - 2 g[0]) + g[0]    -1 ≤ t ≤ 0
           t dg[0] + g[0] + t^3 (dg[0] + dg[1] + 2 g[0] - 2 g[1]) + t^2 (-2 dg[0] - dg[1] - 3 g[0] + 3 g[1])       0 ≤ t ≤ 1
```

In[11]:=
```
(* arrange to form p[t_]:=
  g[-1] φ1[t]+dp[-1] φ2[t]+g[0] φ3[t]+dp[0] φ4[t]+
   g[1] φ5[t]+dp[1] φ6[t]  *)

sss = {ss[[1, 1, 1]], ss[[1, 2, 1]]}
```

Out[11]=
```
{t dg[0] + t^2 (dg[-1] + 2 dg[0] + 3 g[-1] - 3 g[0]) +
  t^3 (dg[-1] + dg[0] + 2 g[-1] - 2 g[0]) + g[0], t dg[0] + g[0] +
  t^3 (dg[0] + dg[1] + 2 g[0] - 2 g[1]) + t^2 (-2 dg[0] - dg[1] - 3 g[0] + 3 g[1])}
```

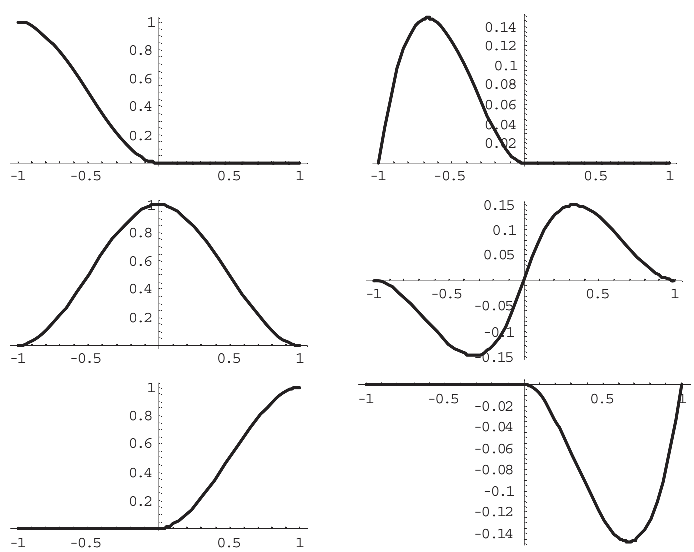
In[12]:=
```
fi = Table[Map[Coefficient[sss, #] &, {g[k - 2], dg[k - 2]}], {k, 1, 3}] //
  Flatten[#, 1] &
```

Out[12]=
```
{{3 t^2 + 2 t^3, 0}, {t^2 + t^3, 0}, {1 - 3 t^2 - 2 t^3, 1 - 3 t^2 + 2 t^3},
 {t + 2 t^2 + t^3, t - 2 t^2 + t^3}, {0, 3 t^2 - 2 t^3}, {0, -t^2 + t^3}}
```

In[13]:=
```
Table[φk[t_] = Piecewise[{{fi[[k, 1]], -1 ≤ t ≤ 0}, {fi[[k, 2]], 0 < t ≤ 1}}],
 {k, 1, 6}]
```

Out[13]=
```
{ 3 t^2 + 2 t^3    -1 ≤ t ≤ 0,   t^2 + t^3    -1 ≤ t ≤ 0,   1 - 3 t^2 - 2 t^3   -1 ≤ t ≤ 0
                                                             1 - 3 t^2 + 2 t^3   0 < t ≤ 1 ,
  t + 2 t^2 + t^3   -1 ≤ t ≤ 0    0               -1 ≤ t ≤ 0    0             -1 ≤ t ≤ 0
  t - 2 t^2 + t^3   0 < t ≤ 1 ,   3 t^2 - 2 t^3   0 < t ≤ 1 ,   -t^2 + t^3    0 < t ≤ 1 }
```

In[14]:=
```
drawing = Table[Plot[φk[t], {t, -1, 1}, AspectRatio → 1 / 2, PlotStyle → Thickness[0.011],
    DisplayFunction → Identity], {k, 1, 6}];
Show[GraphicsArray[Partition[drawing, 2]], DisplayFunction → $DisplayFunction]
```



Out[15]=
```
- GraphicsArray -
```

Appendix: B. The Lebesgue integral

For purposes of this text, it is only necessary to acquire a simple understanding of the definition of this integral and some of its fundamental properties.

The Lebesgue integral for bounded measurable functions: Let $f(x)$ be a (real-valued), bounded measurable function on a closed interval $[a,b]$. Since $f(x)$ is *measurable* on $[a,b]$, there exists a sequence of continuous functions $\{g_n(x)\}$ which converges to $f(x)$ *almost everywhere* on $[a,b]$. Since $f(x)$ is *bounded*, there is a number $K > 0$ for which $|f(x)| \le K$ on the interval $a \le x \le b$. Then we may write

$$\int_a^b f(x)dx = \lim_{n\to\infty}\left(\int_a^b g_n(x)dx\right),$$

where the integrals on the right-hand side are Riemann integrals of continuous functions.

We note, that this definition is correct, because it can be shown that the limit does not depend on the choice of sequence $\{g_n(x)\}$.

Nonnegative integrable functions: Let $f(x) \ge 0$, $a \le x \le b$ be a *measurable* function. Define the sequence of *bounded* measurable functions

$$f_n(x) = \begin{cases} f(x) & x \in [a,b] \text{ and } 0 \le f(x) \le n \\ n & x \in [a,b] \text{ and } n < f(x) \end{cases} \qquad (n = 1,2,\ldots).$$

The function $f(x)$ is said to be **(Lebesgue) integrable** on interval $[a,b]$, if the sequence of integrals $\left\{\int_a^b f_n(x)dx\right\}$ is bounded from above. Then we may write

$$\int_a^b f(x)dx = \lim_{n\to\infty}\int_a^b f_n(x)dx \left(= \sup_n \int_a^b f_n(x)dx\right)$$

and this expression is called **Lebesgue integral** of the function $f(x)$ on $[a,b]$.

Integrable functions of arbitrary signs: Let $f(x)$ be a (real-valued), *measurable* function on a closed interval $[a,b]$. If $f_1(x)$ and $f_2(x)$ are two nonnegative integrable functions such that

$$f(x) = f_1(x) - f_2(x) \qquad (x \in [a,b]),$$

then we say that $f(x)$ is **(Lebesgue) integrable** on $[a,b]$. The number

$$\int_a^b f(x)dx = \int_a^b f_1(x)dx - \int_a^b f_2(x)dx$$

is called the (Lebesgue) **integral** of function $f(x)$ on interval $[a,b]$.

Some properties of the Lebesgue integral and the Riemann integral are the same. For example, the set of integrable functions is a linear space and the map

$$f \to \int_a^b f(x)dx$$

is linear, that is,

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx \quad \text{and} \quad \int_a^b (\lambda f(x))dx = \lambda \int_a^b f(x)dx .$$

In Section 9 we defined the set of (Lebesgue) measure zero. Sets of arbitrary Lebesgue measures can be defined using their *characteristic functions*. If $A=[a,b]$, then the function

$$\chi_A = \begin{cases} 0 \text{ if } & x \notin A \\ 1 \text{ if } & x \in A \end{cases} \qquad (a \le x \le b)$$

is called the *characteristic function* of A.

Measurable sets. A set $A \subseteq [a,b]$ is said to be *measurable* (Lebesgue measurable) if its characteristics function χ_A is integrable. The number

$$\int_a^b \chi_A(x)dx$$

is called the (Lebesgue *measure*) of A and is denoted $\operatorname{mes} A$.

E.B.1. If $A=[c,d] \subseteq [a,b]$, then $\operatorname{mes} A = d-c$. Consequently the Lebesgue measure is a generalization of the length in elementary geometry.

Theorem B.1. If $A \subseteq [a,b]$ and $B \subseteq [a,b]$ are two measurable disjoint sets on the interval $[a,b]$, that is, if $A \cap B = \Theta$, then $A \cup B$ is measurable and

$$\operatorname{mes}(A \cup B) = \operatorname{mes} A + \operatorname{mes} B.$$

Till now we have supposed that the interval $[a,b]$ is arbitrary, but fixed. It can be shown, that the previous definitions and theorems are independent from the choice of such an interval. There are general bounded Lebesgue measurable sets and integrable functions on bounded intervals. The extension of the ideas above to functions which are defined on unbounded intervals can be made if the limit process is applied. Nonnegative integrable functions can be obtained using the following properties:

$$f:(-\infty,+\infty)\to\Re,\ f(x)\geq 0 \quad (x\in\Re),$$

$$f \text{ integrable on every interval } [-n,n] \qquad (n=1,2,\dots),$$

$$\text{the } \lim_{n\to\infty}\int_{-n}^{n} f(x)dx \text{ exists.}$$

The last limit is denoted by $\int_{-\infty}^{\infty} f(x)dx$.

Now it is possible to extend the definition of integrable functions of arbitrary signs to interval $(-\infty,+\infty)$, and the unbounded measurable sets can be discussed in an analogous way as the bounded measurable sets.

The theory of Lebesgue integral in $\Re^1$ can be easily extended to the n-dimensional space $\Re^n$.

Also arbitrary measurable sets may be applied as domains of definition of integrable functions.

Finally, we mention some very important theorems:

Theorem B.2. Let $f(x)$ be an integrable function on the interval $[a,b]$ and let

$$\int_a^b |f(x)|\, dx = 0,$$

then $f(x)=0$ *almost everywhere* on $[a,b]$.

Theorem B.3. (B. Levi)**:** Suppose $\{f_n(x)\}$ is a sequence of non-decreasing non-negative integrable functions on $[a,b]$ and let there be a real number $K>0$ independent from n such that $\int_a^b f_n(x)dx \leq K$ $(n=1,2,\dots)$. Then the function $f(x)=\lim_{n\to\infty} f_n(x)$ is integrable on $[a,b]$ and

$$\int_a^b f(x)dx = \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx = \lim_{n\to\infty} \int_a^b f_n(x)dx .$$

Theorem B.4. (H. Lebesgue)**:** Suppose $\{f_n(x)\}$ is a sequence of integrable functions on $[a,b]$ and $f_n(x)$ converges to $f(x)$ pointwise almost everywhere (that is, except on a set of measure zero). If there is an integrable function $g(x)$ so that for every $n \geq N, |f_n(x)| \leq g(x)$, the pointwise limit $f(x)$ is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx .$$

In other words, the limit process and integration may be interchanged without harm.

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