CONSTRAINT SET TRACKING: OPTIMAL CONTROL PROBLEM

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ABSTRACT. A system of ordinary differential equations subject to additional equality constraints is considered. An optimal control problem and necessary condition are possed for tracking the constraint set

1. INTRODUCTION

There is interest in systems of ordinary differential equations acting on a set constrained by a system of equality constraints.

Let n and m be positive integers. Let y denote a vector in $\mathbb{R}^n$ and z a vector in $\mathbb{R}^m$. The pair (y, z) is called a state vector. Also f is a smooth function mapping $\mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ and g is a smooth function mapping $\mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$.

Let t denote a real number, and for vector valued functions of t, $(\ )'$ denotes the derivative with respect to, wrt, t. State trajectories $(y(t), z(t))$ satisfying the differential equation

$$y'(t) = f(y(t), z(t)) \tag{1.1}$$

and the constraint

$$g(y(t), z(t)) = 0 \tag{1.2}$$

are of interest.

Let an initial state (y_0, z_0) be given that satisfies constraint (1.2). Trajectories leaving the initial state and having a minimum deviation from the constraint set are sought. To find such trajectories an optimal control problem is defined in section 2. A necessary condition for solution of this problem is formulated in section 3. First some definitions are required.

Let u denote a vector in $\mathbb{R}^m$. Such vectors are called controls and for some control trajectory $u(t)$ the following differential equation is introduced.

$$z'(t) = u(t) \tag{1.3}$$

The transpose is denoted by $(\ )^\tau$. The Jacobian of g wrt z is defined by

$$g_z = \begin{pmatrix} g_{1,z_1} & \cdots & g_{1,z_m} \\ \vdots & \ddots & \vdots \\ g_{m,z_1} & \cdots & g_{m,z_m} \end{pmatrix}$$

with g_{i,z_j} denoting the partial derivative of g_i wrt z_j. Similar definitions for g_y, f_z, and f_y.

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Let the error function, $e(y,z,u)$, be defined by

$$e(y,z,u) = g_y(y,z)\cdot f(y,z) + g_z(y,z)\cdot u \tag{1.4}$$

If a state trajectory remains on the constraint set then $g(y(t),z(t)) = 0$ for all t. The derivative wrt t is also zero so

$$\begin{aligned} g'(y(t),z(t)) = \\ g_y(y(t),z(t))\cdot f(y(t),z(t)) + g_z(y(t),z(t))\cdot u(t) = \\ e(y(t),z(t),u(t)) = 0 \end{aligned} \tag{1.5}$$

2. Optimal Control Problem

A control trajectory is sought for generating a state trajectory starting from the initial state having minimum deviation from the constraint set. This can be posed as the following optimal control problem.

For some positive value of t, denoted by t_1, find the control trajectory $u(t)$ defined for $0 \le t \le t_1$ that minimizes the functional $J(u(t))$ defined by

$$J(u(t)) = \int_0^{t_1} e^\tau(y(t),z(t),u(t))\cdot e(y(t),z(t),u(t))dt \tag{2.1}$$

subject to the initial conditions

$$\begin{aligned} y(0) &= y_0 \\ z(0) &= z_0 \end{aligned} \tag{2.2}$$

and state equations (1.1) and (1.3).

3. Necessary Conditions

Let λ denote a vector in $\mathbb{R}^n$ and ξ a vector in $\mathbb{R}^m$. The pair (λ,ξ) is called a costate vector. A Hamiltonian function H is defined by

$$H(y,z,u,\lambda,\xi) = e^\tau(y,z,u)\cdot e(y,z,u) + \lambda^\tau\cdot f(y,z) + \xi^\tau\cdot u \tag{3.1}$$

If control trajectory $u(t)$ is a solution of the optimal control problem it is necessary that a costate trajectory $(\lambda(t),\xi(t))$ exist satisfying the costate equations

$$\lambda'(t) = -H_y^\tau(y(t),z(t),u(t),\lambda(t),\xi(t)) \tag{3.2}$$

$$\xi'(t) = -H_z^\tau(y(t),z(t),u(t),\lambda(t),\xi(t)) \tag{3.3}$$

and end conditions

$$\lambda(t_1) = 0 \tag{3.4}$$

$$\xi(t_1) = 0 \tag{3.5}$$

and the control $u(t)$ gives a minimum of $H(y(t),z(t),u,\lambda(t),\xi(t))$.

The partials of H are defined by

$$H_y(y,z,u,\lambda,\xi) = 2e^\tau(y,z,u)\cdot e_y(y,z,u) + \lambda^\tau\cdot f_y(y,z) \tag{3.6}$$

$$H_z(y,z,u,\lambda,\xi) = 2e^\tau(y,z,u)\cdot e_z(y,z,u) + \lambda^\tau\cdot f_z(y,z) \tag{3.7}$$

The ordered set consisting of a state trajectory, costate trajectory, and control trajectory satisfying the constraints of the optimal control problem and its necessary conditions is called an extremal.

4. Expansion of Necessary Conditions

Since $u(t)$ gives a minimum of H, it is necessary that $H_u(y(t), z(t), u, \lambda(t), \xi(t)) = 0$. Differentiating H defined by (3.1) gives

$$H_u = 2f^\tau \cdot g_y^\tau \cdot g_z + 2u^\tau \cdot g_z^\tau \cdot g_z + \xi^\tau$$

and solving $H_u = 0$ for u gives

$$u = -g_z^{-1} \cdot \left(g_y \cdot f + \frac{1}{2} g_z^{-\tau} \cdot \xi \right) \tag{4.1}$$

where $g_z^{-\tau}$ denotes the transpose of g_z^{-1}.

With $u(t)$ defined by (4.1) $g'(y(t), z(t))$, equation (1.5), becomes

$$g'(y(t), z(t)) = -\frac{1}{2} g_z^{-\tau}(y(t), z(t)) \cdot \xi(t) = e(y(t), z(t), \xi(t)) \tag{4.2}$$

This gives a simplified expression for the error function. Using this expression the partials of e are given by

$$e_y(y, z, \xi) = g_z^{-\tau}(y, z) \cdot \bigotimes_{i=1}^{n} \left(g_z^\tau\right)_{y_i} (y, z) \cdot \left(-\frac{1}{2} g_z^{-\tau}(y, z) \cdot \xi \right) \tag{4.3}$$

$$e_z(y, z, \xi) = g_z^{-\tau}(y, z) \cdot \bigotimes_{i=1}^{m} \left(g_z^\tau\right)_{z_i} (y, z) \cdot \left(-\frac{1}{2} g_z^{-\tau}(y, z) \cdot \xi \right) \tag{4.4}$$

with $\left(g_z^\tau\right)_{y_i} (y, z)$ denoting the partial derivative of $g_z^\tau(y, z)$ wrt y_i, and similarly for $\left(g_z^\tau\right)_{z_i} (y, z)$

For an initial state on the constrained set, (4.2) makes it clear that

$$\xi(0) = 0 \tag{4.5}$$

Also (4.2) suggests using the deviation from the constraint set as a state variable. In particular replace z by a new state vector $\hat{g}$, with $\hat{g} \in \mathbb{R}^m$ and let $z = \hat{z}(y, \hat{g})$ with the function $\hat{z}$ defined implicitly by the relation

$$\hat{g} = g(y, \hat{z}(y, \hat{g})) \tag{4.6}$$

Replace the state equations by

$$y'(t) = f(y(t), \hat{z}(y(t), \hat{g}(t))) \tag{4.7}$$

and

$$\hat{g}'(t) = -\frac{1}{2} g_z^{-\tau}(y(t), \hat{z}(y(t), \hat{g}(t))) \cdot \xi(t) \tag{4.8}$$

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