APPROACH TO CONSTRAINT SET: SIMPLE OPTIMAL CONTROL PROBLEM

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ABSTRACT. A system of ordinary differential equations subject to additional equality constraints is considered. An optimal control problem and necessary condition are possed for approach to the constraint set.

1. INTRODUCTION

There is interest in systems of ordinary differential equations acting on a set constrained by a system of equality constraints. An initial point satisfying the constraints may not be available. In this case trajectories that approach the constrained set must be found. Such trajectories are extremals of a suitable optimal control problem.

Let n and m be positive integers. Let y denote a vector in $\mathbb{R}^n$ and z a vector in $\mathbb{R}^m$. The pair (y, z) is called a state vector. Also f is a smooth function mapping $\mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ and g is a smooth function mapping $\mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$.

Let t denote a real number, and for vector valued functions of t, $(\ )'$ denotes the derivative with respect to, wrt, t. State trajectories $(y(t), z(t))$ satisfying the differential equation

$$y'(t) = f(y(t), z(t)) \tag{1.0.1}$$

and the constraint

$$g(y(t), z(t)) = 0 \tag{1.0.2}$$

are of interest.

Starting from some initial state (y_0, z_0) that does not satisfy constraint (1.0.2), a trajectory approaching the constraint is desired. To find such trajectories an optimal control problem is defined in section 2. A necessary condition for solution of this problem is formulated in section 3. First some definitions are required.

Let U denote the unit cube in $\mathbb{R}^m$. In particular U is defined by

$$U = \{u \in \mathbb{R}^m \ \ni \ -1 \leq u_i \leq 1 \text{ for } i = \ldots m\} \tag{1.0.3}$$

Members of U are called controls and for some control trajectory $u(t)$ the following differential equation is introduced.

$$z'(t) = u(t) \tag{1.0.4}$$

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The transpose is denoted by $(\ )^\tau$. The Jacobian of g wrt z is defined by

$$g_z = \begin{pmatrix} g_{1,z_1} & \cdots & g_{1,z_m} \\ \vdots & \ddots & \vdots \\ g_{m,z_1} & \cdots & g_{m,z_m} \end{pmatrix}$$

with g_{i,z_j} denoting the partial derivative of g_i wrt z_j. Similar definitions for g_y, f_z, and f_y.

2. Optimal Control Problem

A control trajectory is sought for generating a state trajectory starting from the initial state that approachs the constraint set. This can be possed as the following optimal control problem.

For some positive value of t, denoted by t_1, find the control trajectory $u(t)$ with $u(t) \in U$ defined for $0 \leq t \leq t_1$ that minimizes the functional $J(u(t))$ defined by

$$J(u(t)) = \int_0^{t_1} g^\tau(y(t), z(t)) \cdot g(y(t), z(t)) dt \tag{2.0.5}$$

subject to the initial conditions

$$\begin{aligned} y(0) &= y_0 \\ z(0) &= z_0 \end{aligned} \tag{2.0.6}$$

and differential equations (1.0.1) and (1.0.4).

3. Necessary Conditions

Let λ denote a vector in $\mathbb{R}^n$ and ξ a vector in $\mathbb{R}^m$. The pair (λ, ξ) is called a costate vector. A Hamiltonian function H is defined by

$$H(y, z, u, \lambda, \xi) = g^\tau(y, z) \cdot g(y, z) + \lambda^\tau \cdot f(y, z) + \xi^\tau \cdot u \tag{3.0.7}$$

If control trajectory $u(t)$ is a solution of the optimal control problem it is necessary that a costate trajectory $(\lambda(t), \xi(t))$ exist satisfying the differential equations

$$\begin{aligned} \lambda'(t) &= -H_y^\tau(y(t), z(t), u(t), \lambda(t), \xi(t)) \\ \xi'(t) &= -H_z^\tau(y(t), z(t), u(t), \lambda(t), \xi(t)) \end{aligned} \tag{3.0.8}$$

and end conditions

$$\begin{aligned} \lambda(t_1) &= 0 \\ \xi(t_1) &= 0 \end{aligned} \tag{3.0.9}$$

and the control $u(t)$ gives a minimum of $H(y(t), z(t), u, \lambda(t), \xi(t))$ among all controls $u \in U$.

The ordered set consisting of a state trajectory, costate trajectory, and control trajectory satisfying the constraints of the optimal control problem and its necessary conditions is called an extremal.

Since the Hamiltonian H is linear wrt u, minimizing determines $u(t)$ by

$$u(t) = -sign(\xi(t)) \tag{3.0.10}$$

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