TARCER - A Mathematica program for the reduction of two-loop propagator integrals

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Submitted to Computer Physics Communications

Abstract

TARCER is an implementation of the recurrence algorithm of O.V. Tarasov for the reduction of two-loop propagator integrals with arbitrary masses to a small set of basis integrals. The tensor integral reduction scheme is adapted to moment integrals emerging in operator matrix element calculations.

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1 Program Summary

Title of program: TARCER

Version number: 1.0

Available at `http://www.mertig.com/tarcer` *or*
`http://www.physik.uni-leipzig.de/TET/tarcer`

Programming Language: Mathematica 3.0

Platform: Any platform supporting *Mathematica* 3.0

Keywords: Feynman integrals, two-loop corrections

Nature of physical problem: Reduction of Feynman integrals in pertubative two-loop calculations

Method of solution: The system of recurrence relations given by Tarasov [1]

Limitations: Rank of integrals

2 Introduction

TARCER reduces two-loop propagator integrals with arbitrary masses to simpler basis integrals using the reduction algorithm proposed by Tarasov [1,2]. The reduction of scalar products in the numerator of the integral is extended to provide for the presence of an additional external vector that is lightlike. For the reduction of scalar integrals TARCER contains the complete set of recurrence relations given in [1] and some additions for particular parameter configurations. In some cases the number of basic integrals is reduced. Further additions may easily be added.

Due to a vast number of interrelations between the integrals considered it is not immediately obvious how to extract a set of recurrence relations that reduce the complexity of the integrals at each step such that one finally arrives at only a small set of basic integrals [3]. This was achieved by Tarasov. The resulting

[3] The fact that in principle all integrals considered must be expressible in terms of a *finite* set of basic integrals may be seen as follows [3]: Interpret the corresponding Feynman parameter integral representations as integrals over projective differential forms. All forms in question exist on the same differential manifold. De Rham co-

relations are in some cases quite involved. Therefore transcribing them into a program is an error prone process. But, whereas the extraction of adequate relations proves to be quite difficult they can be checked in most cases much more easily. For four or less propagators this may be done via a Mellin-Barnes representation of the integral. In the present program this is automatized in order to minimize input errors.

The emphasis of TARCER is not so much on the speed of evaluation. Our main purpose was to use TARCER to check complicated moment integrals emerging from operator matrix element calculations for small individual moments ($m \leq 5$). An example of this type of integral is given in section 7. Conversely, those integrals, calculated independently by other means, served as a check for TARCER.

The massless case was checked using a FORM program [4]. The reader may also want to compare this program with other existing fully analytic two-loop programs like [5], [6] and [7,8] or the original implementation in [1].

3 Notation

TARCER reduces the following general type of integrals to basic integrals. The *Mathematica* notation, making all dependencies explicit, is listed first:

$$\begin{aligned}&\mathtt{TFI[d,p^2,\Delta p,\{a,b\},\{u,v,r,s,t\},\{\{\nu_1,m_1\},\{\{\nu_2,m_2\},\ldots,\{\nu_5,m_5\}\}]} =\\ &\frac{1}{\pi^d}\int\!\!\int \frac{d^dk_1 d^dk_2\ \ (\Delta k_1)^a\,(\Delta k_2)^b\,(k_1^2)^u\,(k_2^2)^v\,(pk_1)^r\,(pk_2)^s\,(k_1k_2)^t}{[k_1^2-m_1^2]^{\nu_1}\,[k_2^2-m_2^2]^{\nu_2}\,[k_3^2-m_3^2]^{\nu_3}\,[k_4^2-m_4^2]^{\nu_4}\,[k_5^2-m_5^2]^{\nu_5}}\,,\end{aligned} \tag{1}$$

with the abbreviations $k_3 = k_1 - p$, $k_4 = k_2 - p$ and $k_5 = k_1 - k_2$. The exponents $a, \ldots, t$ and the indices $\nu_1, \ldots, \nu_5$ are assumed to be nonnegative integers. Δ denotes a lightlike vector with $\Delta^2 = 0$.

If some of the subsets $\{a, b\}$ or $\{u, v, r, s, t\}$ of exponents vanish we have the following reduced notations:

$$\begin{aligned}&\mathtt{TFI[d,p^2,\Delta p,\{0,0\},\{u,v,r,s,t\},\{\{\nu_1,m_1\},\{\nu_2,m_2\},\ldots,\{\nu_5,m_5\}\}]} =\\ &\mathtt{TFI[d,p^2,\{u,v,r,s,t\},\{\{\nu_1,m_1\},\{\nu_2,m_2\},\ldots,\{\nu_5,m_5\}\}]}\,,\end{aligned} \tag{2}$$

$$\begin{aligned}&\mathtt{TFI[d,p^2,\Delta p,\{a,b\},\{0,0,0,0,0\},\{\{\nu_1,m_1\},\{\nu_2,m_2\},\ldots,\{\nu_5,m_5\}\}]} =\\ &\mathtt{TFI[d,p^2,\Delta p,\{a,b\},\{\{\nu_1,m_1\},\{\nu_2,m_2\},\ldots,\{\nu_5,m_5\}\}]}\,,\end{aligned} \tag{3}$$

homology then implies that there exists only a finite number of inequivalent forms on this manifold. This in turn implies the above statement.

$$\texttt{TFI}[\texttt{d},\texttt{p}^2,\Delta\texttt{p},\{0,0\},\{0,0,0,0,0\},\{\{\nu_1,\texttt{m}_1\},\{\nu_2,\texttt{m}_2\},\ldots,\{\nu_5,\texttt{m}_5\}\}] =$$
$$\texttt{TFI}[\texttt{d},\texttt{p}^2,\{\{\nu_1,\texttt{m}_1\},\{\nu_2,\texttt{m}_2\},\ldots,\{\nu_5,\texttt{m}_5\}\}]\ . \tag{4}$$

Furthermore, if a mass vanishes the argument $\{\nu_j, 0\}$ of TFI may be replaced by the index ν_j alone. For example the fully massless integral with five proagators may be entered as $\texttt{TFI}[\texttt{d},\texttt{p}^2,\{1,1,1,1,1\}]$.

In the course of evaluation all scalar products in the numerator will be eliminated. If no more scalar products are present we keep the notation in line with [1]:

$$\texttt{TFI}[\texttt{d},\texttt{p}^2,\{\{\nu_1,\texttt{m}_1\},\{\nu_2,\texttt{m}_2\},\ldots,\{\nu_5,\texttt{m}_5\}\}] =$$
$$F^{(d)}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \frac{1}{\pi^d}\int\!\!\int \frac{d^dk_1 d^dk_2}{[k_1^2-m_1^2]^{\nu_1}\ [k_2^2-m_2^2]^{\nu_2}\ \cdots [k_5^2-m_5^2]^{\nu_5}}\ , \tag{5}$$

$$\texttt{TVI}[\texttt{d},\texttt{p}^2,\{\{\nu_1,\texttt{m}_1\},\{\nu_2,\texttt{m}_2\},\{\nu_3,\texttt{m}_3\},\{\nu_4,\texttt{m}_4\}\}] =$$
$$V^{(d)}_{\nu_1\nu_2\nu_3\nu_4} = \frac{1}{\pi^d}\int\!\!\int \frac{d^dk_1 d^dk_2}{[k_5^2-m_1^2]^{\nu_1}\ [k_2^2-m_2^2]^{\nu_2}\ [k_3^2-m_3^2]^{\nu_3}\ [k_4^2-m_4^2]^{\nu_4}}\ , \tag{6}$$

$$\texttt{TJI}[\texttt{d},\texttt{p}^2,\{\{\nu_1,\texttt{m}_1\},\{\nu_2,\texttt{m}_2\},\{\nu_3,\texttt{m}_3\}\}] =$$
$$J^{(d)}_{\nu_1\nu_2\nu_3} = \frac{1}{\pi^d}\int\!\!\int \frac{d^dk_1 d^dk_2}{[k_1^2-m_1^2]^{\nu_1}\ [k_5^2-m_2^2]^{\nu_2}\ [k_4^2-m_3^2]^{\nu_3}}\ , \tag{7}$$

$$\texttt{TJI}[\texttt{d},0,\{\{\nu_1,\texttt{m}_1\},\{\nu_2,\texttt{m}_2\},\{\nu_3,\texttt{m}_3\}\}] =$$
$$K^{(d)}_{\nu_1\nu_2\nu_3} = \frac{1}{\pi^d}\int\!\!\int \frac{d^dk_1 d^dk_2}{[k_1^2-m_1^2]^{\nu_1}\ [k_5^2-m_2^2]^{\nu_2}\ [k_2^2-m_3^2]^{\nu_3}} \tag{8}$$

and

$$\texttt{TBI}[\texttt{d},\texttt{p}^2,\{\{\nu_1,\texttt{m}_1\},\{\nu_2,\texttt{m}_2\}\}] =$$
$$B^{(d)}_{\nu_1\nu_2} = \frac{1}{\pi^{d/2}}\int \frac{d^dk_1}{[k_1^2-m_1^2]^{\nu_1}\ [k_3^2-m_2^2]^{\nu_2}}\ , \tag{9}$$

$$\texttt{TAI}[\texttt{d},0,\{\{\nu_1,\texttt{m}_1\}\}] =$$
$$A^{(d)}_{\nu_1} = \frac{1}{\pi^{d/2}}\int \frac{d^dk_1}{[k_1^2-m_1^2]^{\nu_1}}\ . \tag{10}$$

Input for TARCER should be prepared in terms of the TFI-notation.

4 Elimination of numerators

The TARCER-function `TarcerRecurse` is the principal function which, when applied to an expression involving the above integrals, performs the complete reduction to the set of basic integrals. In the first step the numerator of the integrand in the `TFI`-integrals is simplified as far as possible by standard manipulations until an irreducible numerator of the form $(\Delta k_1)^a\,(\Delta k_2)^b\,(pk_1)^r\,(pk_2)^s$ results. The remaining integrals are of the form

$$I^{(d)}_{abrs} = \mathtt{TFI}[\mathtt{d}, \mathtt{p}^2, \Delta\mathtt{p}, \{\mathtt{a}, \mathtt{b}\}, \{\mathtt{0}, \mathtt{0}, \mathtt{r}, \mathtt{s}, \mathtt{0}\}, \{\{\nu_1, \mathtt{m}_1\}, \ldots, \{\nu_5, \mathtt{m}_5\}\}]\;. \quad (11)$$

The critical observation of Tarasov is that integrals containing irreducible numerators may be rewritten in terms of scalar integrals in higher space-time dimensions and that those can later be reduced again to scalar integrals in the original space-time dimension. Integrals of the form (11) we rewrite as

$$I^{(d)}_{abrs} = T_{abrs}(p^2, \Delta p, \{\partial\}, \mathbf{d}^+)\; I^{(d)}_{0000}\;, \quad (12)$$

by employing an operator T that is a polynomial in the operator $\mathbf{d}^+$ representing a shift $d \to d+2$ in dimension and in the mass derivatives $\partial_j = \partial/\partial m_j^2$. With the lightlike vector Δ present, the required T-operator is given by a generalization of eq. (25) of [1] [4]:

$$\begin{aligned} T_{abrs}(p^2, \Delta p, \{\partial\}, \mathbf{d}^+) &= \left(\frac{\partial}{i\partial\gamma_1}\right)^a \left(\frac{\partial}{i\partial\gamma_2}\right)^b \left(\frac{\partial}{i\partial\beta_1}\right)^r \left(\frac{\partial}{i\partial\beta_2}\right)^s \\ \times \exp\Big[i\Big\{&(\beta_1 p^2 + \gamma_1(\Delta p))\,Q_1 + (\beta_2 p^2 + \gamma_2(\Delta p))\,Q_2 \\ &+\beta_1(\beta_1 p^2 + 2\gamma_1(\Delta p))\,Q_{11} + \beta_2(\beta_2 p^2 + 2\gamma_2(\Delta p))\,Q_{22} \\ &+(\beta_1\beta_2 p^2 + (\beta_1\gamma_2 + \beta_2\gamma_1)(\Delta p))\,Q_{12}\Big\}\,\rho\Big]\Bigg|_{\substack{\beta_i=0,\,\gamma_i=0\\ \rho=-\mathbf{d}^+}}\;, \end{aligned} \quad (13)$$

where the Q's are polynomials in ∂_j

$$\begin{aligned} 4i\,Q_{11} &= \partial_2 + \partial_4 + \partial_5\;,\; -Q_1 = \partial_3\partial_5 + \partial_4\partial_5 + \partial_2\partial_3 + \partial_3\partial_4\;, \\ 4i\,Q_{22} &= \partial_1 + \partial_3 + \partial_5\;,\; -Q_2 = \partial_4\partial_5 + \partial_3\partial_5 + \partial_1\partial_4 + \partial_3\partial_4\;, \\ 2i\,Q_{12} &= \partial_5\;. \end{aligned} \quad (14)$$

[4] We have $\rho = -\mathbf{d}^+$ instead of $-1/\pi^2\,\mathbf{d}^+$ due to the factor $1/\pi^d$ in the normalization of the integrals in (12).

5 Recurrence relations

Once all irreducible numerators are eliminated the next step taken by the function `TarcerRecurse` is to repeatedly apply the recurrence relations that reduce the exponents of the scalar propagators in the integrals until no further reduction is possible. All recurrence relations explicitly or implicitly given by Tarasov are implemented.

For certain classes of on-shell integrals we found it necessary to supplement the recurrence relations by some additions in order to achieve maximal reduction. Those additional relations are listed in the following. The relations obtained by permutations of indices and masses are also implemented. The operators $\mathbf{1}^{\pm}, \mathbf{2}^{\pm}, \ldots$ act on an integral by in-/decreasing the first, second, etc. index by one unit.

For $m_1 = 0$ and $\nu_2 > 1$:

$$\begin{aligned} &2\ m_2^2\ (1-\nu_2)\ \nu_2\ \mathbf{2}^+ J^{(d)}_{\nu_1 \nu_2 \nu_3} = \\ &\qquad (d-2\ \nu_2)\ (1-\nu_2)\ J^{(d)}_{\nu_1 \nu_2 \nu_3} - \nu_1\ (-d+2\ \nu_1+2)\ \mathbf{1}^+\mathbf{2}^- J^{(d)}_{\nu_1 \nu_2 \nu_3}\ . \end{aligned} \tag{15}$$

For $p^2 = m_1^2$, $m_2 = m_3 = 0$ and $\nu_1 > 0$:

$$\begin{aligned} &\nu_1\ (d-2\ \nu_2-2\ \nu_3)\ (d-\nu_2-\nu_3-1)\ \mathbf{1}^+ J^{(d)}_{\nu_1 \nu_2 \nu_3} = \\ &\qquad -\nu_2\ (-d+2\ \nu_2+2)\ (-2\ d+\nu_1+2\ \nu_2+2\ \nu_3+2)\mathbf{2}^+ J^{(d)}_{\nu_1 \nu_2 \nu_3}\ . \end{aligned} \tag{16}$$

For $m_1 = m_2 = 0$, $p^2 = m_3^2$ and $\nu_1 > 1$:

$$\begin{aligned} J^{(d)}_{\nu_1 \nu_2 \nu_3} = &-\frac{(d-\nu_1-\nu_2)\ (-2-d+2\,\nu_1+2\,\nu_2)}{2\,{m_3}^2\ (d-2\,\nu_1)\ (-1+\nu_1)} \\ &\times\ \frac{(3\,d-2\,\nu_1-2\,\nu_2-2\,\nu_3)\ (-1-d+\nu_1+\nu_2+\nu_3)}{(2\,d-2\,\nu_1-2\,\nu_2-\nu_3)\ (1+2\,d-2\,\nu_1-2\,\nu_2-\nu_3)}\ \mathbf{1}^- J^{(d)}_{\nu_1 \nu_2 \nu_3}\ . \end{aligned} \tag{17}$$

For $m_1 = 0$, $m_2 = m_3$, $p^2 = 0$ and $\nu_1 > 0$:

$$\begin{aligned} K^{(d)}_{\nu_1 \nu_2 \nu_3} = &\frac{(d-2\ (-1+\nu_1+\nu_2))\ (1+d-\nu_1-\nu_2-\nu_3)}{2\,{m_2}^2\ (d-2\,\nu_1)\ (1+d-2\,\nu_1-\nu_2-\nu_3)} \\ &\times\ \frac{(d-2\ (-1+\nu_1+\nu_3))}{(2+d-2\,\nu_1-\nu_2-\nu_3)}\ \mathbf{1}^-\ K^{(d)}_{\nu_1 \nu_2 \nu_3}\ . \end{aligned} \tag{18}$$

For $m_1 = m_2 = m_3$, $p^2 = m_1^2$ and $\nu_1 > 0$:

$$
\begin{aligned}
&16\ m_1^2\ \nu_1\ (d-\nu_1-\nu_2-\nu_3)\ \mathbf{1}^+ J^{(d)}_{\nu_1\nu_2\nu_3} = \\
&\quad -(1+3\ d-3\ \nu_1-4\ \nu_2)\ \nu_3\ \mathbf{1}^-\mathbf{3}^+ J^{(d)}_{\nu_1\nu_2\nu_3} \\
&\quad -\nu_2\ (1+3\ d-3\ \nu_1-4\ \nu_3)\ \mathbf{1}^-\mathbf{2}^+ J^{(d)}_{\nu_1\nu_2\nu_3} \\
&\quad +(-1+2\ d-\nu_1-2\ \nu_2)\ \nu_3\ \mathbf{2}^-\mathbf{3}^+ J^{(d)}_{\nu_1\nu_2\nu_3} \\
&\quad -\big(-6\ d^2+22\ d\ \nu_1-16\ \nu_1^2-\nu_2+6\ d\ \nu_2-13\ \nu_1\ \nu_2 \\
&\quad -\nu_3+6\ d\ \nu_3-13\ \nu_1\ \nu_3-4\ \nu_2\ \nu_3)\ J^{(d)}_{\nu_1\nu_2\nu_3} \\
&\quad +\nu_2\ (-1+2\ d-\nu_1-2\ \nu_3)\ \mathbf{2}^+\mathbf{3}^-\ J^{(d)}_{\nu_1\nu_2\nu_3}\ .
\end{aligned}
\tag{19}
$$

Note that in the last case all $J^{(d)}_{\nu_1\nu_2\nu_3}$ are eventually reduced to $J^{(d)}_{111}$ and $(A^{(d)}_1)^2$. That is, the number of basic two-loop integrals is smaller than in the general case.

Other special configurations may require still further additional relations.

6 Verification of recurrence relations

As stated in [1] the derivation 'of these relations is rather tedious and for brevity of the presentation will be omitted'. Therefore, we rederived most of the recurrence relations. Furthermore, an automatic verification of recurrence relations for the integrals of type J or V with three and four propagators in the general mass case is possible via the kernel of the respective Mellin-Barnes type integral-representations.

By replacing the massive scalar propagators in the momentum space integral with their respective Mellin-Barnes representation and interchanging integrations, as already described in [9] for the case $\nu_i = 1$, we obtain the Mellin-Barnes representation of J for general $\nu_i > 0$:

$$
J^{(d)}_{\nu_1\nu_2\nu_3} = \frac{1}{(2\pi i)^3} \iiint_{-i\infty}^{+i\infty} \psi^{(d)}_{\nu_1\nu_2\nu_3}(s_1,s_2,s_3)\,(-p^2)^{d-\nu-s} \prod_{i=1}^{3} (m_i^2)^{s_i}\ ds_i\ , \tag{20}
$$

with the kernel

$$
\begin{aligned}
&\psi^{(d)}_{\nu_1\nu_2\nu_3}(s_1,s_2,s_3) = \\
&(-1)^{1+\nu} \frac{\Gamma(\nu+s-d)}{\Gamma(3d/2-\nu-s)} \prod_{i=1}^{3} \frac{\Gamma(-s_i)\Gamma(d/2-\nu_i-s_i)}{\Gamma(\nu_i)}\quad ,
\end{aligned}
\tag{21}
$$

where $\nu = \nu_1+\nu_2+\nu_3$ and $s = s_1+s_2+s_3$. As usual the integration contours separate the poles of the gamma functions in the left and right half planes.

The Mellin-Barnes representation of V contains four s-integrations in an analogous manner with the kernel

$$
\begin{aligned}
&\psi^{(d)}_{\nu_1\nu_2\nu_3\nu_4}(s_1,s_2,s_3,s_4) = \qquad (22)\\
&(-1)^{1+\nu}\frac{\Gamma(\nu_1+\nu_3+s_1+s_3-d/2)\Gamma(d-\nu_1-\nu_3-\nu_4-s_1-s_3-s_4)}{\Gamma(\nu_1+\nu_3+\nu_4+s_1+s_3+s_4-d/2)\Gamma(d-\nu_1-\nu_3-s_1-s_3)}\\
&\qquad\times\frac{\Gamma(\nu+s-d)}{\Gamma(3d/2-\nu-s)}\frac{\Gamma(-s_4)\Gamma(\nu_4+s_4)}{\Gamma(\nu_4)}\prod_{i=1}^{3}\frac{\Gamma(-s_i)\Gamma(d/2-\nu_i-s_i)}{\Gamma(\nu_i)}\ ,
\end{aligned}
$$

where $\nu=\nu_1+\nu_2+\nu_3+\nu_4$ and $s=s_1+s_2+s_3+s_4$.

Insertion of the above integral representation for J into a vanishing combination of J's as provided by a conjectured recurrence relation yields under the integral a sum of the form

$$
\sum_{n_1,n_2,n_3}\phi_{n_1n_2n_3}(s_1,s_2,s_3)\,(-p^2)^{-n-s}(m_1^2)^{s_1+n_1}(m_2^2)^{s_2+n_2}(m_3^2)^{s_3+n_3}\ , \quad (23)
$$

where the n_i's run over a finite set of integers and $n=n_1+n_2+n_3$. Here each ϕ represents a sum over ψ's with various values of the indices ν_i and d. By an appropriate shift of integration contours for each term separately accompanied by a change of variables $s_i\to s_i-n_i$ one extracts a global factor:

$$
(-p^2)^{-s}(m_1^2)^{s_1}(m_2^2)^{s_2}(m_3^2)^{s_3}\sum_{n_1,n_2,n_3}\phi_{n_1n_2n_3}(s_1-n_1,s_2-n_2,s_3-n_3)\,. \quad (24)
$$

The sum in (24) uniquely determines the coefficients of the large p^2 expansion of the Mellin-Barnes integral over (24) in terms of (fractional) powers of $m_i^2/(-p^2)$. Therefore, if the recurrence relation is indeed correct and this integral vanishes then the above sum has to vanish identically.

This observation provides a method to verify recurrence relations for both J and V, where a similar Ansatz applies. The method is suitable for recurrence relations that are valid for general values of m_i^2 and p^2. It can also be adapted to the case when some of the m_i^2 or p^2 are zero. It is, however, not applicable to relations that hold only for particular parameter configurations, e.g., when some kinematical determinants vanish.

This verification-procedure is implemented in TARCER through the functions `CheckTJIRecursion` and `CheckTVIRecursion`. The notebook `TARCER.nb` also contains comments for each particular recursion as to whether this check is applicable or not.

Despite the complexity of the equations presented in [1] we found only one non-obvious [5] misprint: In eq. (67) the global sign of the right hand side has to be reversed.

7 Usage and examples

The *Mathematica* 3.0 notebook `TARCER.nb` contains the complete source code. Upon evaluation the recurrence relations are incorporated into the function `TarcerRecurse`. All T-operators (13) up to $\{a+b, r+s\} \leq$ `$RankLimit` are constructed and the corresponding relations (12) explicitly generated. The value of `$RankLimit` can be set in the prologue section.

The evaluation of `TARCER.nb` generates a binary file `tarcer.mx` containing all functions and definitions in internal *Mathematica* format. An explicit integral reduction only requires the latter to be loaded directly into *Mathematica* 3.0. A set of preproduced tarcer*.mx files for different operating systems is also available. Therefore one only needs to run the `TARCER.nb` notebook if one wants to increase `$RankLimit` or wishes to modify the program.

`TarcerRecurse` is the main function that performs the integral reduction. Its usage is simply as follows: It can be applied to any expression involving the functions `TFI, TVI, TJI, TBI` and `TAI` whose first argument is a symbol `d`.

For a simple example, consider the following on-shell integral where $p^2 = M^2$:

$$\frac{1}{\pi^d}\int\!\!\int \frac{d^dk_1 d^dk_2}{k_1^2\;[k_2^2-M^2]\;[(k_1-p)^2-M^2]\;(k_2-p)^2\;[(k_1-k_2)^2-M^2]}\;. \tag{25}$$

Application of `TarcerRecurse` to the corresponding input form (4) yields:

$$\mathtt{TarcerRecurse[TFI[d,M^2,\{1,\{1,M\},\{1,M\},1,\{1,M\}\}]]}$$

$$-\frac{3\;(d-2)^2\;(5\;d-18)\;\left(\mathbf{A}^{(d)}_{\{1,M\}}\right)^2}{32\;(d-4)^2\;(d-3)\;M^6}+\frac{(3\;d-10)\;(3\;d-8)\;\mathbf{J}^{(d)}_{\{1,M\},\{1,0\},\{1,0\}}}{8\;(d-4)\;(2\;d-7)\;M^4}+\frac{(3d-10)\;(3d-8)\;\mathbf{J}^{(d)}_{\{1,M\},\{1,M\},\{1,M\}}}{16\;(d-4)^2\;M^4} \tag{26}$$

[5] The lower limit of the sum in (46) should read $j=1$ and m_3 in (66) needs to be replaced by m_3^2.

Note that the display format of TARCER explicitly shows the masses together with the indices.

The function `TarcerExpand` inserts explicit results for some basis integrals as specified by the option `TarcerReduce`. A second argument to `TarcerExpand` must be given in form of a rule, like $\mathtt{d} \to 4 + \varepsilon$. Then an expansion of the first argument of `TarcerExpand` in the sole variable specified, here ε, around 0 will be performed.

Applying `TarcerExpand` to `%`, i.e. to the previous expression, yields:

$$\mathtt{TarcerExpand}[\%, \mathtt{d} \to 4 + \varepsilon]$$

$$(M^2)^{\varepsilon-1} S_\varepsilon^2 \left(6\ \zeta(2)\ \log(2) - \frac{3\ \zeta(3)}{2}\right) \tag{27}$$

Here $S_\varepsilon = e^{\gamma_E\ (d-4)/2}$ is used.[6] This result is of course well known, see [10] where the expansion up to $\mathcal{O}(\varepsilon^3)$ is given.

As a somewhat more demanding example consider another on-shell integral with $p^2 = M^2$ containing Δk_1 in the numerator:

$$\frac{1}{\pi^d} \int\int \frac{d^d k_1 d^d k_2\ \ (\Delta k_1)^m}{[k_1^2 - M^2]^2\ k_2^2\ (k_1 - p)^2\ [(k_2 - p)^2 - M^2]\ [(k_1 - k_2)^2 - M^2]} \,. \tag{28}$$

For general m one can, with some effort, express this integral in terms of a moment integral, Laurent-expanded around $d = 4$:

$$\begin{aligned}
(M^2)^{d-6}\ (\Delta p)^m\ S_\varepsilon^2 \int_0^1 dx\ x^m \Bigg(& \left(\frac{1}{2} + \frac{3}{8}\,\zeta(2) - \frac{1}{2\ (d-4)^2}\right) \delta(1-x) + \\
& \Bigg(-\zeta(2) + \frac{\log(1-x)}{2\ (1-x)} - \frac{1}{4} \log^2(1-x) - \frac{\log(x)}{2\ (1+x)} - \frac{\log(x)}{2\ (1-x)^2} + \\
& \left(1 - \frac{1}{(1-x)^2}\right)\left(\frac{1}{2}\ \log(x)\ \log(1-x) + \frac{3}{4}\log^2(x)\right) - \\
& \left(1 - \frac{2}{(1-x)^2}\right)\left(\frac{1}{2}\ \zeta(2) + \log(x)\ \log(1+x) + \mathrm{Li}_2(-x)\right) + \\
& \left(1 - \frac{1}{2\ (1-x)^2}\right) \mathrm{Li}_2(1-x)\Bigg) + \mathcal{O}(d-4)\Bigg)\ .
\end{aligned} \tag{29}$$

[6] γ_E is the Euler-constant

TARCER was originally written to check this kind of general expressions for individual moments. Consider for example the moment $m = 1$ and apply `TarcerRecurse` to the input form (3):

$$\texttt{TarcerRecurse[TFI[d,}\,\mathtt{M}^2,\Delta\mathtt{p},\{1,0\},\{\{2,\mathtt{M}\},1,1,\{1,\mathtt{M}\},\{1,\mathtt{M}\}\}]]$$

$$-\frac{(d-2)^2\ (27\ d^4-404\ d^3+2175\ d^2-4902\ d+3776)\ \Delta p\ \left(\mathbf{A}^{(d)}_{\{1,M\}}\right)^2}{256\ (d-5)^2\ (d-4)^2\ (d-3)\ M^8}-$$
$$\frac{(d-2)\ (3\ d-10)\ (3\ d-8)\ \Delta p\ \mathbf{J}^{(d)}_{\{1,M\},\{1,0\},\{1,0\}}}{32\ (d-5)\ (d-4)\ (2d-7)\ M^6}+$$
$$\frac{d\ (3\ d-10)\ (3\ d-8)\ \Delta p\ \mathbf{J}^{(d)}_{\{1,M\},\{1,M\},\{1,M\}}}{128\ (d-4)^2\ M^6} \tag{30}$$

Expansion with `TarcerExpand` yields:

$$\texttt{TarcerExpand[\%, d}\to 4+\varepsilon]$$

$$(M^2)^{\varepsilon-2}\Delta p\ S_\varepsilon^2\left(3\ \log(2)\ \zeta(2)-\frac{11\ \zeta(2)}{8}-\frac{1}{2\ \varepsilon^2}-\frac{3\ \zeta(3)}{4}+\frac{1}{2}\right) \tag{31}$$

Which agrees with the analytic integration of the moment-integral above for $m = 1$.

For a massive off-shell example consider the integral

$$\frac{1}{\pi^d}\int\!\!\int\frac{d^dk_1d^dk_2\ \ (k_1^2)^2}{[k_2^2-m_4^2]\ [(k_1-p)^2-m_2^2]\ [(k_1-k_2)^2-m_3^2]}\ , \tag{32}$$

denoted Y^{11}_{234} in [8]. Acting with `TarcerRecurse` on the input form (2) yields:

$$\texttt{TarcerRecurse[TFI[d,}\,\mathtt{p}^2,\{2,0,0,0,0\},\{0,\{1,\mathtt{m}_4\},\{1,\mathtt{m}_2\},0,\{1,\mathtt{m}_3\}\}]]$$

$$\frac{(d\ (p^2+m_2^2+m_3^2+7\ m_4^2)-12\ m_4^2)\ \mathbf{A}^{(d)}_{\{1,m_2\}}\ \mathbf{A}^{(d)}_{\{1,m_3\}}}{3\ (3\ d-4)}+$$
$$\frac{(d\ (p^2+m_2^2+7\ m_3^2+m_4^2)-12\ m_3^2)\ \mathbf{A}^{(d)}_{\{1,m_2\}}\ \mathbf{A}^{(d)}_{\{1,m_4\}}}{3\ (3\ d-4)}+$$
$$\frac{((7\ d-12)\ m_2^2+d\ (p^2+m_3^2+m_4^2))\ \mathbf{A}^{(d)}_{\{1,m_3\}}\ \mathbf{A}^{(d)}_{\{1,m_4\}}}{3\ (3\ d-4)}+$$

$$
\begin{aligned}
&\frac{1}{3\ (d-2)\ (3\ d-4)}(((d-2)\ d\ p^4+2\ (d-4)\ (d-3)\ (m_3^2+m_4^2)\ p^2+\\
&\quad (d\ (25\ d-102)+96)\ m_2^4-2\ (d-3)\ (d\ m_3^4+\\
&\quad 2\ (7\ d-12)\ m_4^2\ m_3^2+d\ m_4^4)+2\ m_2^2\ ((19\ (d-4)\ d+72)\ p^2-\\
&\quad (d-3)\ (5\ d-12)\ (m_3^2+m_4^2)))\ \ \mathbf{J}^{(d)}_{\{1,m_4\}\{1,m_3\}\{1,m_2\}})+\\
&\frac{1}{3\ (d-2)\ (3\ d-4)}(2\ m_3^2\ (-3\ (3\ d-4)\ m_2^4+2\ ((18-11\ d)\ p^2+\\
&\quad 2\ (d-3)\ m_3^2)\ m_2^2-12\ m_4^2\ (-p^2+3\ m_3^2+m_4^2)+\\
&\quad d\ (-p^4-4\ m_4^2\ p^2+m_3^4+9\ m_4^4+22\ m_3^2\ m_4^2))\ \ \mathbf{J}^{(d)}_{\{2,m_3\}\{1,m_4\}\{1,m_2\}})+\\
&\frac{1}{3\ (d-2)\ (3\ d-4)}(2\ m_4^2\ (-3\ (3\ d-4)\ m_2^4+2\ ((18-11\ d)\ p^2+\\
&\quad 2\ (d-3)\ m_4^2)\ m_2^2+3\ (3\ d-4)\ m_3^4+2\ m_3^2\ ((11\ d-18)\ m_4^2-\\
&\quad 2\ (d-3)\ p^2)+d\ (m_4^4-p^4))\ \ \mathbf{J}^{(d)}_{\{2,m_4\}\{1,m_3\}\{1,m_2\}})-\\
&\frac{1}{3\ (d-2)\ (3\ d-4)}(8\ m_2^2\ (m_2^2-p^2)\ ((2\ d-3)\ p^2+(2\ d-3)\ m_2^2-\\
&\quad (d-3)\ (m_3^2+m_4^2))\ \ \mathbf{J}^{(d)}_{\{2,m_2\}\{1,m_4\}\{1,m_3\}}\ )
\end{aligned}
\tag{33}
$$

which is a much shorter expression than the one given by eq. (B.2) in the appendix of [8]. Moreover, no spurious massless propagator is introduced.

These and further examples can also be found in the material on the WWW-sites listed in the program summary.

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