GigaNumerics in Deep Structure Analysis of Images

Employing MathLink and the Parallel Computing Toolkit

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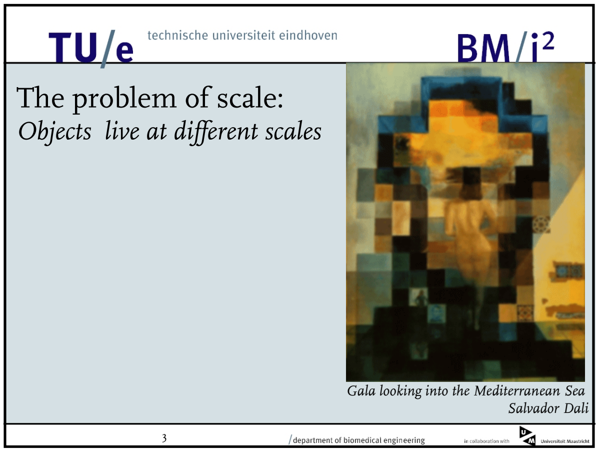
Outline

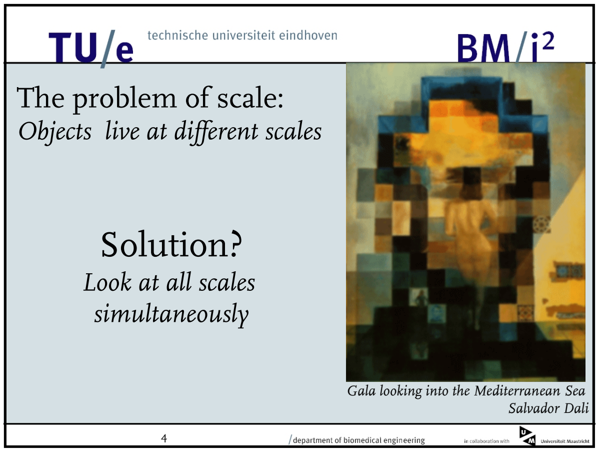
- Scale Space
- Toppoints
- Image Reconstruction
- Mathematica Implementation



The problem of scale:

Objects live at different scales

Gala looking into the Mediterranean Sea
Salvador Dali



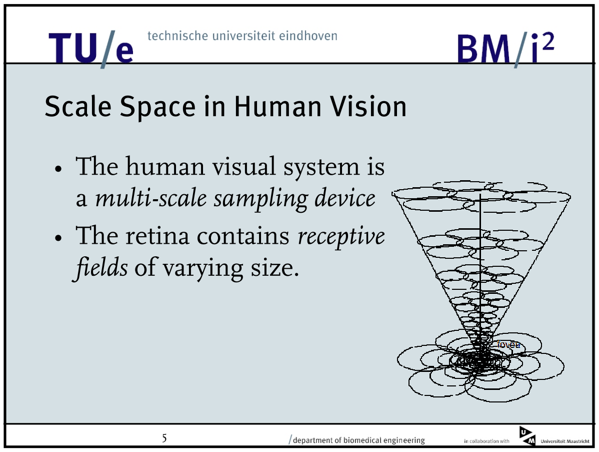
The problem of scale:

Objects live at different scales

Solution?

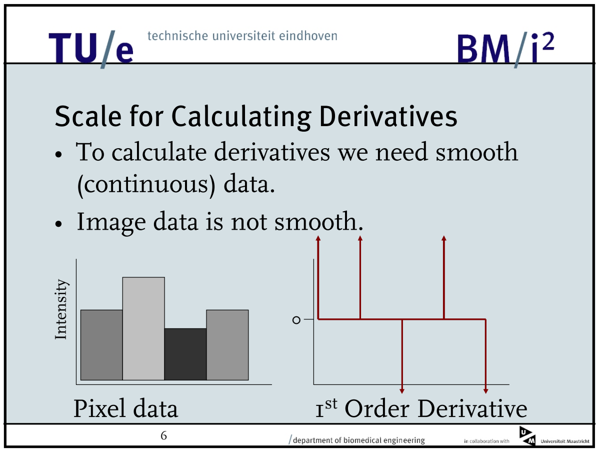
Look at all scales simultaneously

Gala looking into the Mediterranean Sea
Salvador Dali



Scale Space in Human Vision

- The human visual system is a *multi-scale sampling device*
- The retina contains *receptive fields* of varying size.

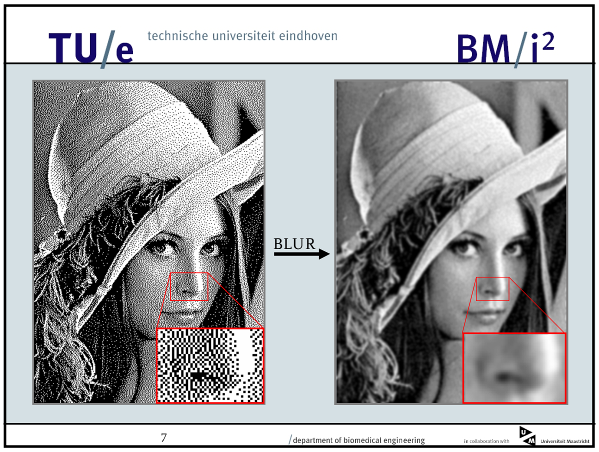


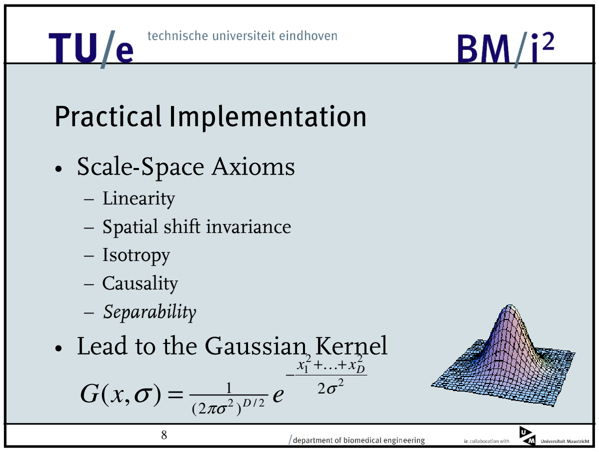
Scale for Calculating Derivatives

- To calculate derivatives we need smooth (continuous) data.
- Image data is not smooth.

Pixel data

1st Order Derivative





Practical Implementation

- Scale-Space Axioms
  - Linearity
  - Spatial shift invariance
  - Isotropy
  - Causality
  - *Separability*
- Lead to the Gaussian Kernel

$$G(x,\sigma) = \frac{1}{(2\pi\sigma^2)^{D/2}} e^{-\frac{x_1^2+\ldots+x_D^2}{2\sigma^2}}$$

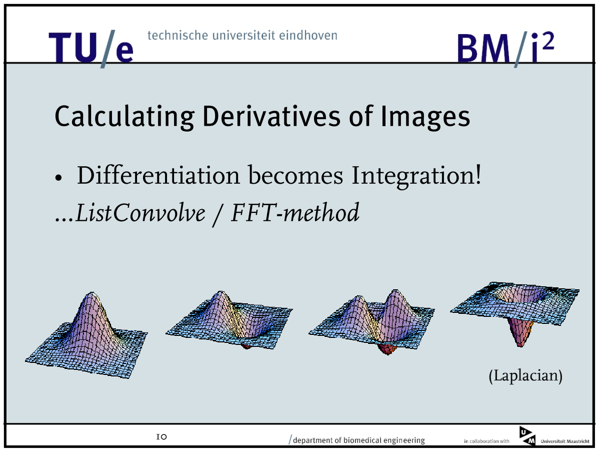
Calculating Derivatives with Gaussians

- The derivative of the data at a scale σ is defined as

$$\frac{\partial}{\partial x_1}\{L_0(x)\otimes G(x;\sigma)\}$$

- Due to nice properties of the Gaussian this can be rewritten as

$$L_0(x)\otimes\frac{\partial}{\partial x_1}G(x;\sigma)$$

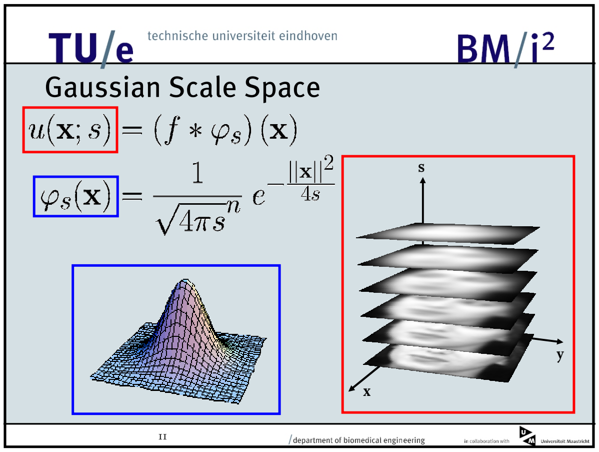


Calculating Derivatives of Images

- Differentiation becomes Integration!

...*ListConvolve / FFT-method*

(Laplacian)



Gaussian Scale Space

$$\boxed{u(\mathbf{x};s)} = (f*\varphi_s)(\mathbf{x})$$

$$\boxed{\varphi_s(\mathbf{x})} = \frac{1}{\sqrt{4\pi s}^{n}} e^{-\frac{||\mathbf{x}||^2}{4s}}$$

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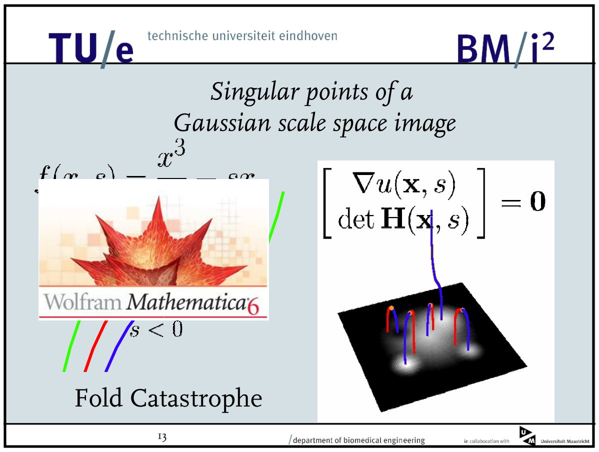
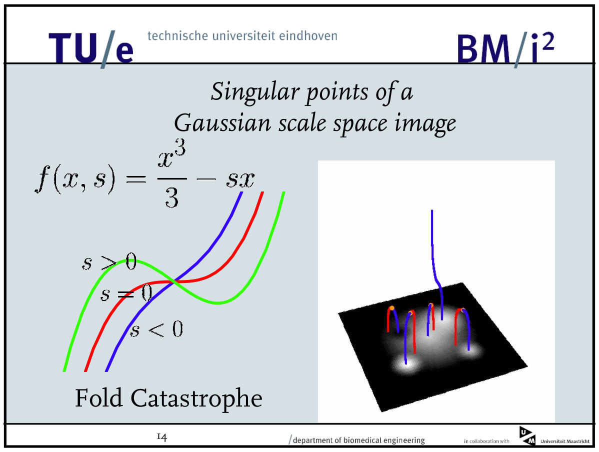


Singular points of a Gaussian scale space image

$$\begin{bmatrix} \nabla u(\mathbf{x}, s) \\ \det \mathbf{H}(\mathbf{x}, s) \end{bmatrix} = \mathbf{0}$$

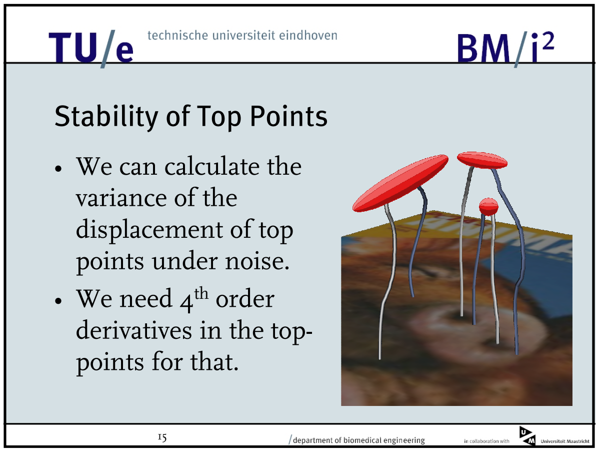
Fold Catastrophe



Singular points of a Gaussian scale space image

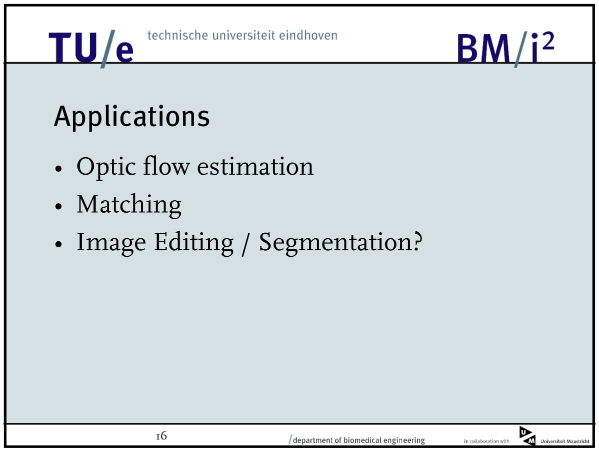
$$f(x, s) = \frac{x^3}{3} - sx$$

Fold Catastrophe



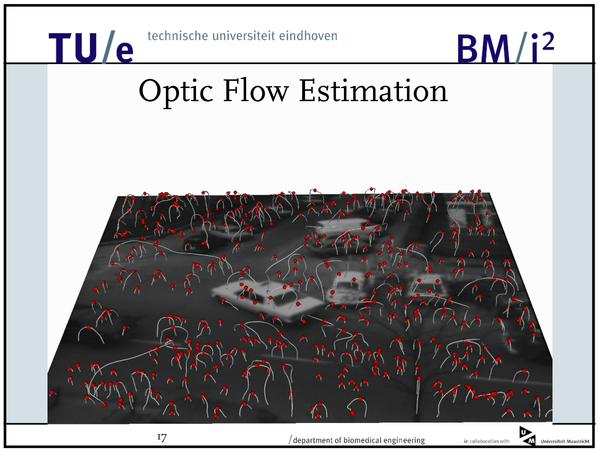
Stability of Top Points

- We can calculate the variance of the displacement of top points under noise.
- We need 4th order derivatives in the top-points for that.

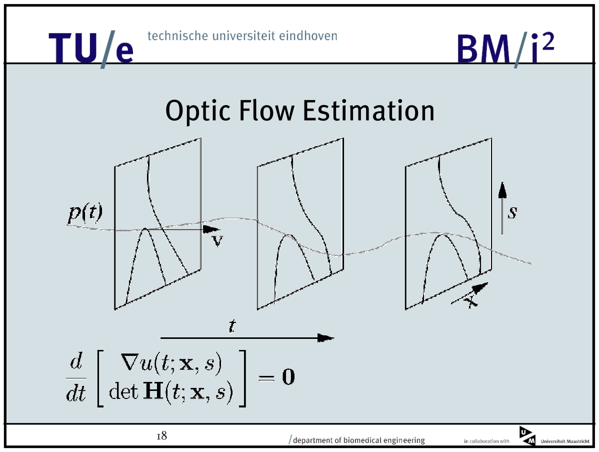


Applications

- Optic flow estimation
- Matching
- Image Editing / Segmentation?

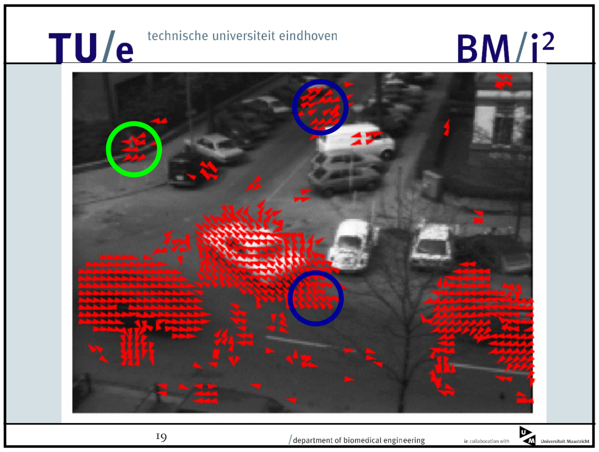


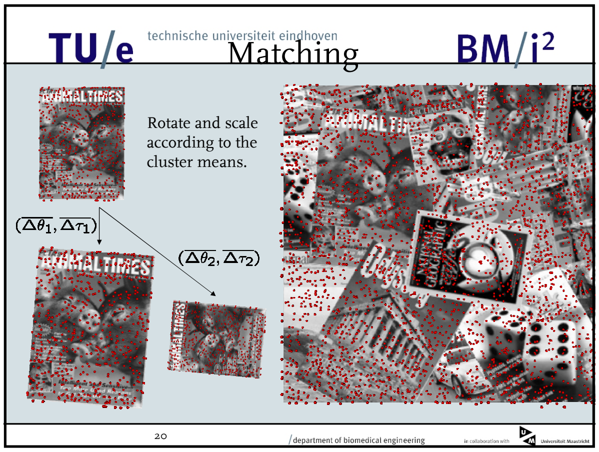
Optic Flow Estimation



Optic Flow Estimation

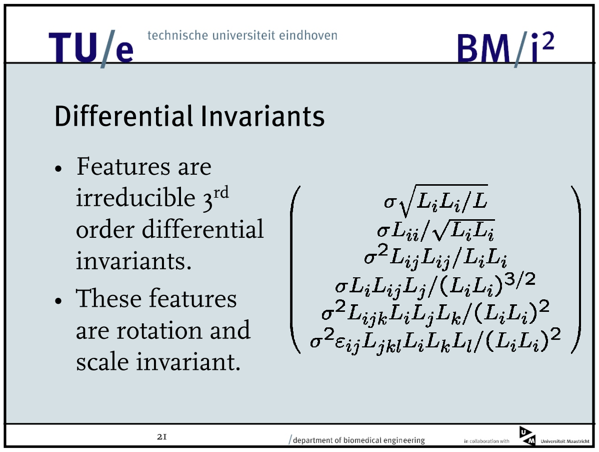
$$\frac{d}{dt}\begin{bmatrix} \nabla u(t; \mathbf{x}, s) \\ \det \mathbf{H}(t; \mathbf{x}, s) \end{bmatrix} = \mathbf{0}$$





Matching

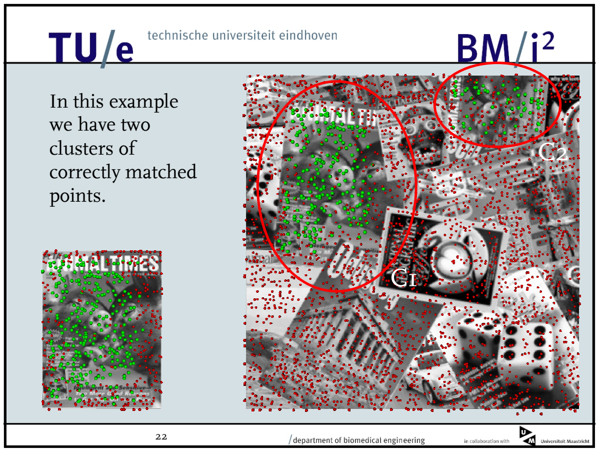
Rotate and scale according to the cluster means.



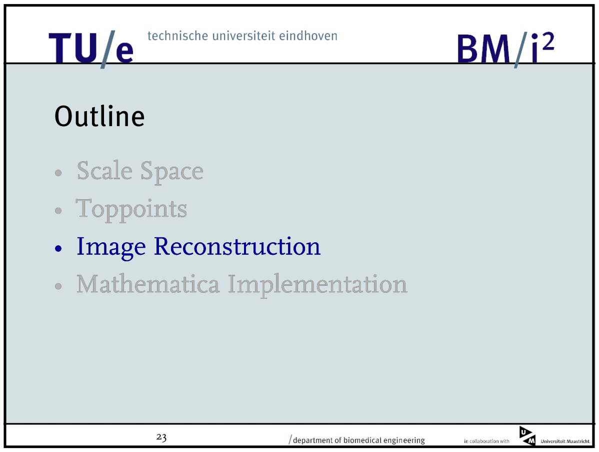
Differential Invariants

- Features are irreducible 3rd order differential invariants.
- These features are rotation and scale invariant.

$$\begin{pmatrix} \sigma\sqrt{L_iL_i/L} \\ \sigma L_{ii}/\sqrt{L_iL_i} \\ \sigma^2 L_{ij}L_{ij}/L_iL_i \\ \sigma L_iL_{ij}L_j/(L_iL_i)^{3/2} \\ \sigma^2 L_{ijk}L_iL_jL_k/(L_iL_i)^2 \\ \sigma^2 \varepsilon_{ij}L_{jkl}L_iL_kL_l/(L_iL_i)^2 \end{pmatrix}$$



In this example we have two clusters of correctly matched points.



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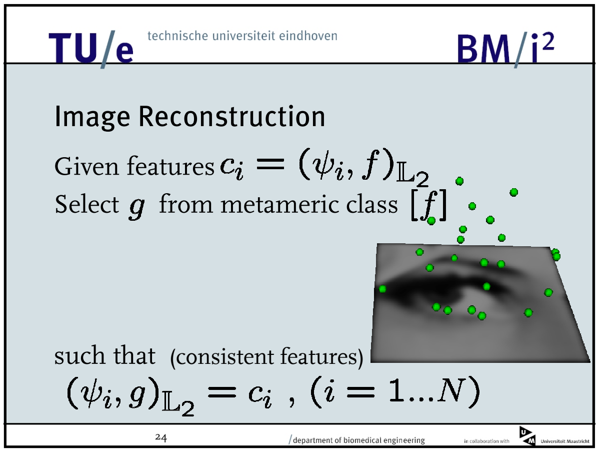


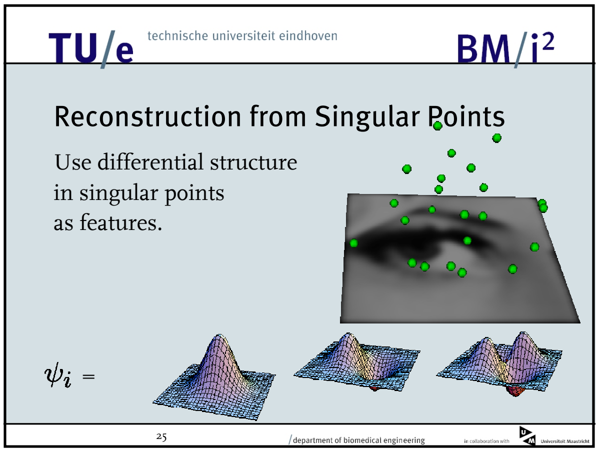
Image Reconstruction

Given features $c_i = (\psi_i, f)_{\mathbb{L}_2}$

Select g from metameric class $[f]$

such that (consistent features)

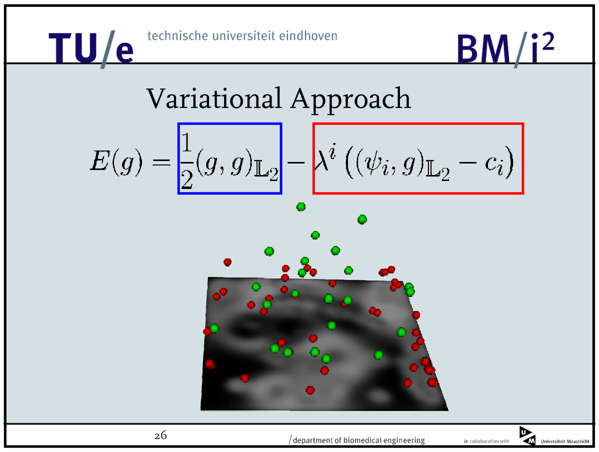
$$(\psi_i, g)_{\mathbb{L}_2} = c_i \;,\; (i = 1 ... N)$$



Reconstruction from Singular Points

Use differential structure
in singular points
as features.

$\psi_i =$



Variational Approach

$$E(g) = \boxed{\frac{1}{2}(g,g)_{\mathbb{L}_2}} - \boxed{\lambda^i\left((\psi_i, g)_{\mathbb{L}_2} - c_i\right)}$$

Minimisation of $E(g) = \frac{1}{2}(g,g)_A$

under the constraints $((\kappa_i, g)_A - c_i) = 0$

$$(f,g)_A = (f,g)_{\mathbb{L}_2} + (Af, Ag)_{\mathbb{L}_2}$$

The solution is an A-orthogonal projection of f onto κ_i

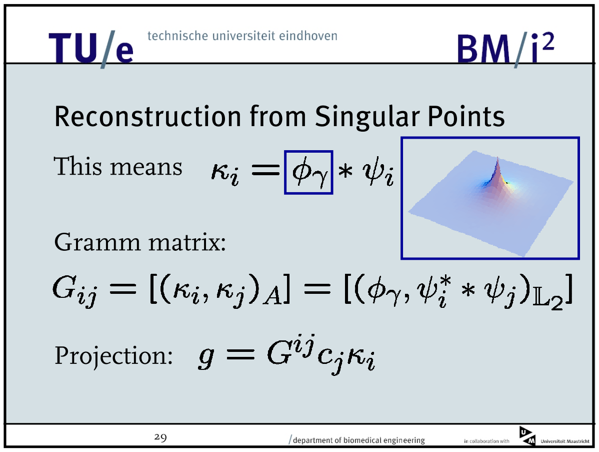
Generalisation using gelfand triples (R. Duits)

Prior and Dual Filters

$$A = -\gamma\sqrt{-\Delta}$$
$$(f,g)_A = (f,g)_{\mathbb{L}_2} + (\gamma\nabla f, \gamma\nabla g)_{\mathbb{L}_2}$$

$$(\kappa_i, f)_A = (\psi_i, f)_{\mathbb{L}_2}$$

$$\kappa_i = (I + A^\dagger A)^{-1}\psi_i$$



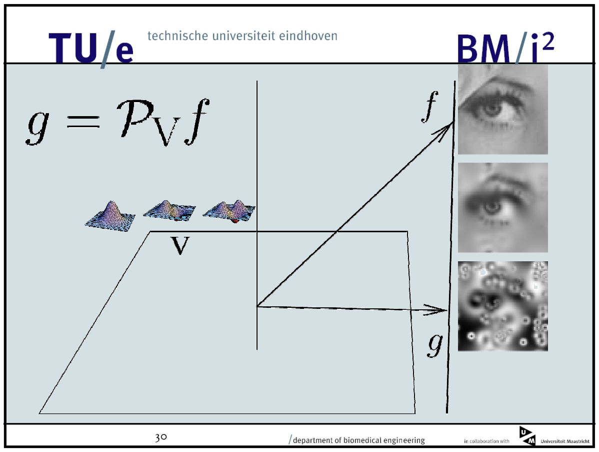
Reconstruction from Singular Points

This means $\kappa_i = \boxed{\phi_\gamma} * \psi_i$

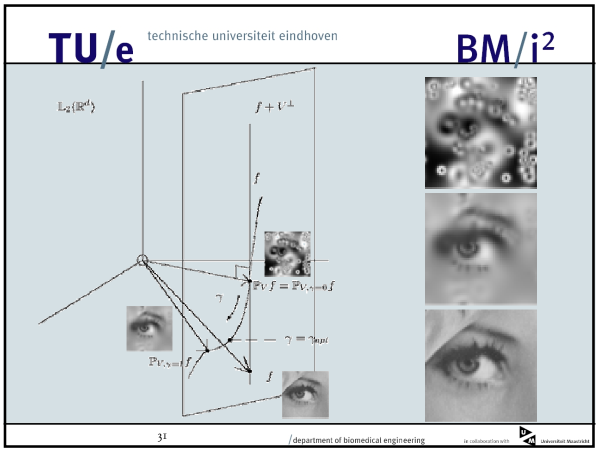
Gramm matrix:

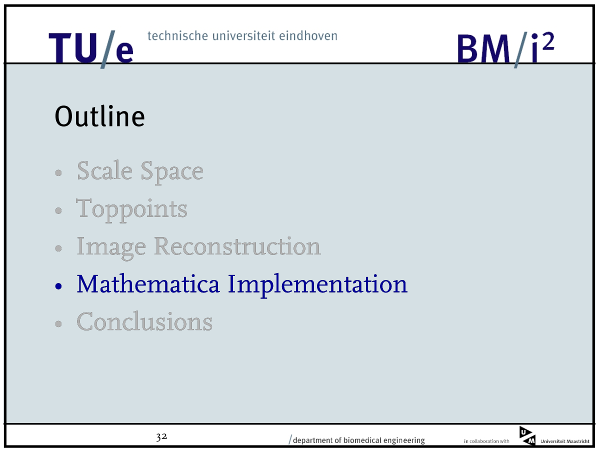
$$G_{ij} = [(\kappa_i, \kappa_j)_A] = [(\phi_\gamma, \psi_i^* * \psi_j)_{\mathbb{L}_2}]$$

Projection: $g = G^{ij} c_j \kappa_i$



$$g = \mathcal{P}_V f$$





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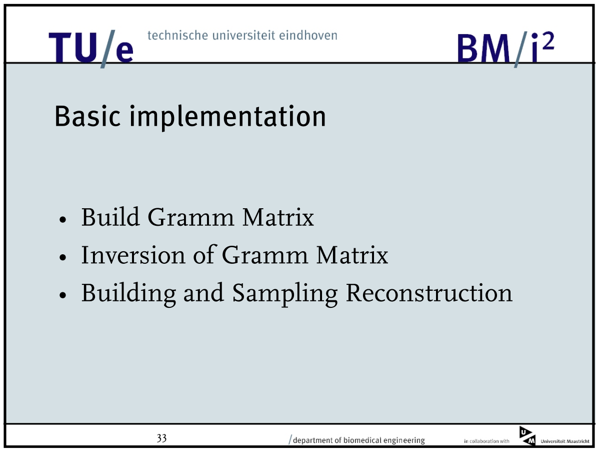
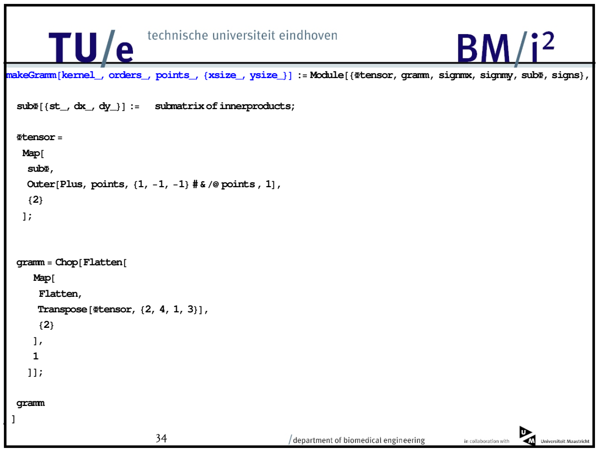
- Conclusions

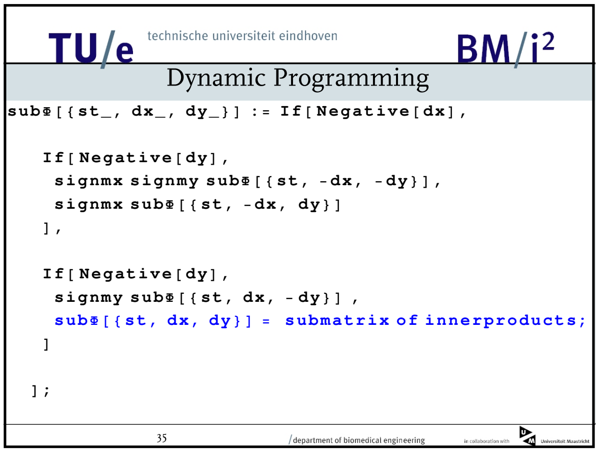


Basic implementation

- Build Gramm Matrix
- Inversion of Gramm Matrix
- Building and Sampling Reconstruction

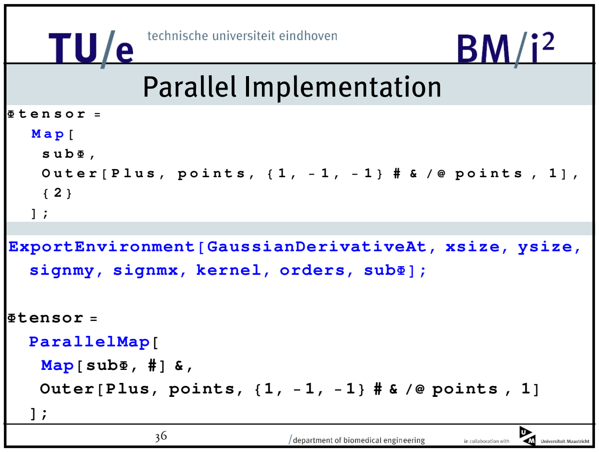


```
makeGramm[kernel_, orders_, points_, {xsize_, ysize_}] := Module[{Φtensor, gramm, signmx, signmy, subΦ, signs},

 subΦ[{st_, dx_, dy_}] :=   submatrix of innerproducts;

 Φtensor =
  Map[
   subΦ,
   Outer[Plus, points, {1, -1, -1} # & /@ points , 1],
   {2}
  ];


 gramm = Chop[Flatten[
    Map[
     Flatten,
     Transpose[Φtensor, {2, 4, 1, 3}],
     {2}
    ],
    1
   ]];

 gramm
]
```



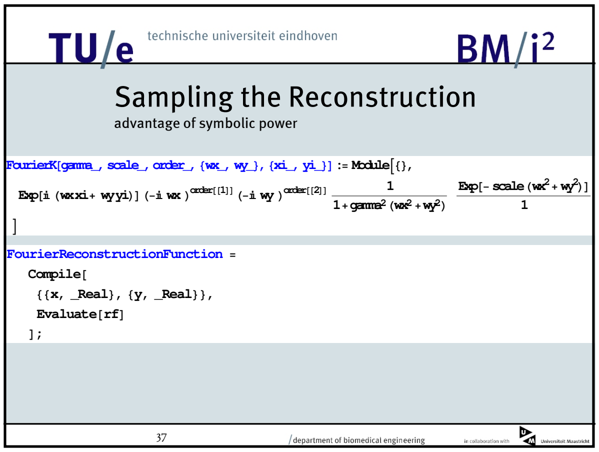
Dynamic Programming

```
subΦ[{st_, dx_, dy_}] := If[Negative[dx],

   If[Negative[dy],
    signmx signmy subΦ[{st, -dx, -dy}],
    signmx subΦ[{st, -dx, dy}]
   ],

   If[Negative[dy],
    signmy subΦ[{st, dx, -dy}] ,
    subΦ[{st, dx, dy}] = submatrix of innerproducts;
   ]

  ];
```



Parallel Implementation

```
Φtensor =
  Map[
   subΦ,
   Outer[Plus, points, {1, -1, -1} # & /@ points , 1],
   {2}
  ];

ExportEnvironment[GaussianDerivativeAt, xsize, ysize,
  signmy, signmx, kernel, orders, subΦ];

Φtensor =
  ParallelMap[
   Map[subΦ, #] &,
   Outer[Plus, points, {1, -1, -1} # & /@ points , 1]
  ];
```

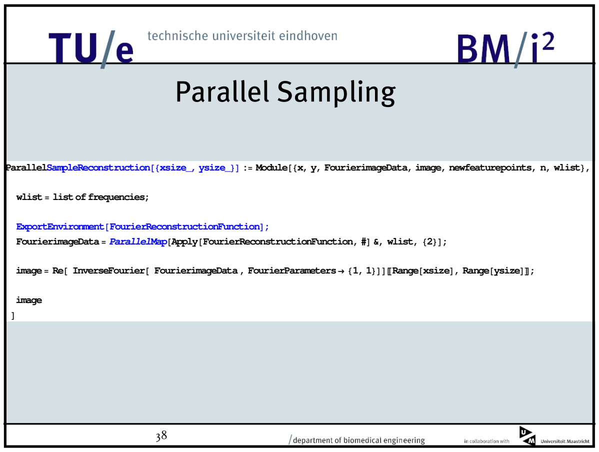


Sampling the Reconstruction

advantage of symbolic power

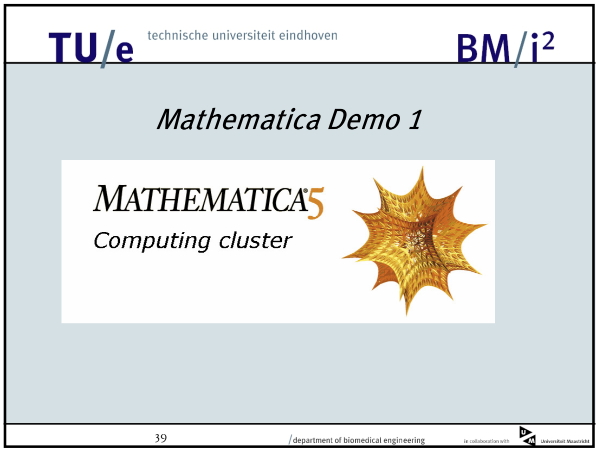
```
FourierK[gamma_, scale_, order_, {wx_, wy_}, {xi_, yi_}] := Module[{},
 Exp[ⅈ (wx xi + wy yi)] (-ⅈ wx )^order[[1]] (-ⅈ wy )^order[[2]] 1/(1 + gamma^2 (wx^2 + wy^2)) Exp[- scale (wx^2 + wy^2)]/1
]
```

```
FourierReconstructionFunction =
   Compile[
    {{x, _Real}, {y, _Real}},
    Evaluate[rf]
   ];
```



Parallel Sampling

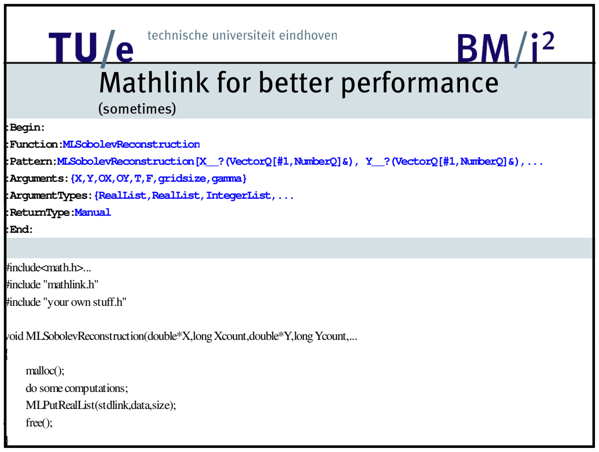
```
ParallelSampleReconstruction[{xsize_, ysize_}] := Module[{x, y, FourierimageData, image, newfeaturepoints, n, wlist},

  wlist = list of frequencies;

  ExportEnvironment[FourierReconstructionFunction];
  FourierimageData = ParallelMap[Apply[FourierReconstructionFunction, #] &, wlist, {2}];

  image = Re[ InverseFourier[ FourierimageData , FourierParameters → {1, 1}]]⟦Range[xsize], Range[ysize]⟧;

  image
 ]
```



Mathematica Demo 1

MATHEMATICA5

Computing cluster

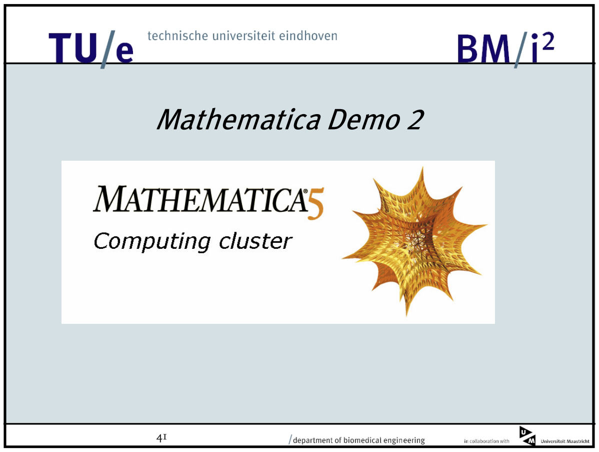


Mathlink for better performance

(sometimes)

```
:Begin:
:Function:MLSobolevReconstruction
:Pattern:MLSobolevReconstruction[X__?(VectorQ[#1,NumberQ]&), Y__?(VectorQ[#1,NumberQ]&),...
:Arguments:{X,Y,OX,OY,T,F,gridsize,gamma}
:ArgumentTypes:{RealList,RealList,IntegerList,...
:ReturnType:Manual
:End:
```

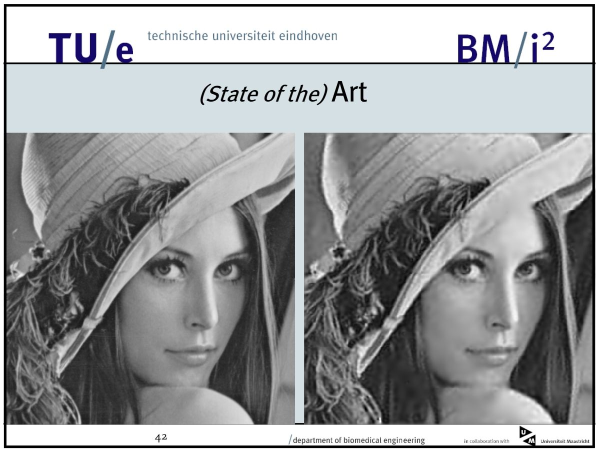
```
#include<math.h>...
#include "mathlink.h"
#include "your own stuff.h"

void MLSobolevReconstruction(double*X,long Xcount,double*Y,long Ycount,...
{
      malloc();
      do some computations;
      MLPutRealList(stdlink,data,size);
      free();
}
```



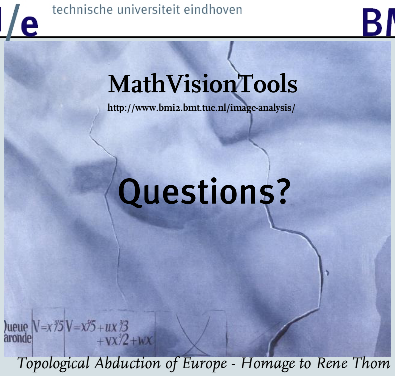
Mathematica Demo 2

MATHEMATICA5

Computing cluster



(State of the) Art



MathVisionTools

http://www.bmi2.bmt.tue.nl/image-analysis/

Questions?

Topological Abduction of Europe - Homage to Rene Thom
Salvador Dali

Acknowledgements

- Bart ter Haar Romeny
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