Invariants and Symmetries for Partial Differential Equations and Lattices*

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Abstract

Methods for the computation of invariants and symmetries of nonlinear evolution, wave, and lattice equations are presented. The algorithms are based on dimensional analysis, and can be implemented in any symbolic language, such as *Mathematica*. Invariants and symmetries are shown for several well-known equations.

Our *Mathematica* package allows one to automatically compute invariants and symmetries. Applied to systems with parameters, the package determines the conditions on these parameters so that a sequence of invariants or symmetries exists. The software can thus be used to test the integrability of model equations for wave phenomena.

1 The Key Concept: Scaling Invariance

The ubiquitous Korteweg-de Vries (KdV) equation from soliton theory,

$$u_t = 6uu_x + u_{3x}, \tag{1}$$

is invariant under the dilation (scaling) symmetry $(t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2 u)$, where λ is an arbitrary parameter. Obviously, u corresponds to two derivatives in x, i.e. $u \sim \partial^2/\partial x^2$. Introducing weights, $w(u) = 2$ if we set $w(\partial/\partial x) = 1$. Similarly, $\partial/\partial t \sim \partial^3/\partial x^3$, thus $w(\partial/\partial t) = 3$. The *rank* of a monomial equals the sum of all of its weights. Observe that (1) is *uniform in rank* since all the terms have rank $R = 5$.

Likewise, the Volterra lattice, which is one of the discretizations of (1),

$$\dot{u}_n = u_n\,(u_{n+1} - u_{n-1}), \tag{2}$$

is invariant under $(t, u_n) \rightarrow (\lambda^{-1}t, \lambda u_n)$. So, $u_n \sim \mathrm{d/dt}$, or $w(u_n) = 1$ if we set $w(\mathrm{d/dt}) = 1$. Every term in (2) has rank $R = 2$, thus (2) is uniform in rank.

Scaling invariance, which is a special Lie-point symmetry, is common to many integrable nonlinear partial differential equations (PDEs) such as (1), and nonlinear differential-difference equations (DDEs) like (2). Both equations have infinitely many polynomial invariants [1, 5] and symmetries [3]. In this paper we show how to use the scaling invariance to explicitly compute polynomial invariants and symmetries of PDEs and DDEs.

2 Computation of Invariants

For PDE systems as in Table 1, the conservation law $\mathrm{D}_t\rho = \mathrm{D}_x J$ connects the *invariant* (conserved density) ρ and the associated *flux* J. As usual, D_t and D_x are total derivatives. Most *polynomial* density-flux pairs only depend on $\mathbf{u}, \mathbf{u}_x$, etc. (not explicitly on t and x). Integration of the conservation law with respect to x yields that $P = \int_{-\infty}^{+\infty} \rho\, dx$ is constant in time, provided J vanishes at infinity. P is a conserved quantity.

*Research supported in part by NSF under Grant CCR-9625421.

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The first three (of infinitely many) conservation laws for (1) are

$$\mathrm{D}_t(u) = \mathrm{D}_x(3u^2 + u_{2x}), \quad \mathrm{D}_t(u^2) = \mathrm{D}_x(4u^3 - u_x^2 + 2uu_{2x}), \tag{3}$$

$$\mathrm{D}_t(u^3 - \frac{1}{2}u_x^2) = \mathrm{D}_x(\frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{2x} + \frac{1}{2}u_{2x}^2 - u_xu_{3x}). \tag{4}$$

The densities $\rho = u, u^2, u^3 - \frac{1}{2}u_x^2$ have ranks 2, 4 and 6, respectively.

Conserved densities of PDEs like (1) can be computed as follows:

- Require that each equation in the PDE system is uniform in rank. Solve the resulting linear system to determine the weights of the dependent variables. For (1), solve $w(u) + w(\partial/\partial t) = 2w(u) + 1 = w(u) + 3$, to get $w(u) = 2$ and $w(\partial/\partial t) = 3$.
- Select the rank R of ρ, say, $R = 6$. Make a linear combination of all the monomials in the components of $\mathbf{u}$ and their x-derivatives that have rank R. Remove 'equivalent' monomials, that is, those that are total x-derivatives (like u_{4x}) or differ by a total x-derivative. For example, uu_{2x} and u_x^2 are equivalent since $uu_{2x} = \frac{1}{2}(u^2)_{2x} - u_x^2$. For (1), one gets $\rho = c_1u^3 + c_2u_x^2$ of rank $R = 6$.
- Substitute ρ into the conservation law, eliminate all t-derivatives, and require that the resulting expression is a total x-derivative. Apply the Euler operator [1] to avoid integration by parts. The remaining part must vanish identically. This yields a linear system for the constants c_i. Solve the system. For (1), one gets $c_1 = 1, c_2 = -1/2$.

See [1] for the complete algorithm and its implementation. See [4] for an integrated *Mathematica* Package that computes invariants (and also symmetries) of PDEs and DDEs.

TABLE 1
Invariants and Symmetries

	Continuous Case (PDEs)	Semi-discrete Case (DDEs)
System	$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$	$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$
Cons. Law	$\mathrm{D}_t\rho = \mathrm{D}_xJ$	$\dot{\rho}_n = J_n - J_{n+1}$
Symmetry	$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u} + \epsilon\mathbf{G})\vert_{\epsilon=0}$	$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u}_n + \epsilon\mathbf{G})\vert_{\epsilon=0}$

For DDEs like (2), compute the weights in a similar way. Determine all monomials of rank R in the components of $\mathbf{u}_n$ and their t-derivatives. Use the DDE to replace all the t-derivatives. Monomials are 'equivalent' if they belong to the same equivalence class of shifted monomials. Keep only the main representatives (centered at n) of the various classes. Combine these representatives linearly with coefficients c_i, and substitute the form of ρ_n into the conservation law $\dot{\rho}_n = J_n - J_{n+1}$. Remove all t-derivatives and pattern-match the resulting expression with $J_n - J_{n+1}$. Set the non-matching part equal to zero, and solve the linear system for the c_i. Determine J_n from the pattern $J_n - J_{n+1}$. For (2), the first three (of infinitely many) densities ρ_n are listed in Table 2. Details about the algorithm and its implementation are in [2, 4, 5].

3 Computation of Symmetries

As summarized in Table 1, $\mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$ is a *symmetry* of a PDE system iff it leaves it invariant for the change $\mathbf{u} \to \mathbf{u} + \epsilon\mathbf{G}$ within order ϵ. Hence, $\mathrm{D}_t(\mathbf{u} + \epsilon\mathbf{G}) = \mathbf{F}(\mathbf{u} + \epsilon\mathbf{G})$ must hold up to order ϵ. Thus, $\mathbf{G}$ must satisfy the linearized equation $\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}]$, where $\mathbf{F}'$ is the Fréchet derivative: $\mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u} + \epsilon\mathbf{G})|_{\epsilon=0}$.

Using the dilation invariance, generalized symmetries $\mathbf{G}$ can be computed as follows:

• Determine the weights of the dependent variables as in Section 2.

• Select the rank R of the symmetry. Make a linear combination of all the monomials involving $\mathbf{u}$ and its x-derivatives of rank R. For example, for (1), $G = c_1\, u^2 u_x + c_2\, u_x u_{2x} + c_3\, u u_{3x} + c_4\, u_{5x}$ is the generalized symmetry of rank $R = 7$.

• Compute $\mathrm{D}_t\mathbf{G}$. Use the PDE system to remove all t-derivatives. Equate the result to the Fréchet derivative $\mathbf{F}'(\mathbf{u})[\mathbf{G}]$. Treat the different monomial terms in $\mathbf{u}$ and its x-derivatives as independent, to get the linear system for c_i. Solve that system. For (1), one obtains

$$G = 30u^2 u_x + 20u_x u_{2x} + 10u u_{3x} + u_{5x}. \tag{5}$$

The symmetries of rank 3, 5, and 7 are listed in Table 2. They are the first three of infinitely many. For DDEs like (2), $\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ is a *symmetry* iff

TABLE 2
Prototypical Examples

	Korteweg-de Vries Equation	Volterra Lattice
Equation	$u_t = 6uu_x + u_{3x}$	$\dot{u}_n = u_n\,(u_{n+1} - u_{n-1})$
Invariants	$\rho = u$ $\quad\rho = u^2$ $\rho = u^3 - \frac{1}{2}u_x^2$	$\rho_n = u_n$ $\quad\rho_n = u_n(\frac{1}{2}u_n + u_{n+1})$ $\rho_n = \frac{1}{3}u_n^3 + u_n u_{n+1}(u_n + u_{n+1} + u_{n+2})$
Symmetries	$\mathbf{G} = u_x$ $\quad\mathbf{G} = 6uu_x + u_{3x}$ $\mathbf{G} = 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}$	$\mathbf{G} = u_n u_{n+1}(u_n + u_{n+1} + u_{n+2})$ $-u_{n-1}u_n(u_{n-2} + u_{n-1} + u_n)$

the infinitesimal transformation $\mathbf{u}_n \to \mathbf{u}_n + \epsilon\mathbf{G}$ leaves the DDE invariant within order ϵ. Consequently, $\mathbf{G}$ must satisfy $\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}]$, where $\mathbf{F}'$ is the Fréchet derivative, $\mathbf{F}'(\mathbf{u}_n)[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u}_n + \epsilon\mathbf{G})|_{\epsilon=0}$.

Algorithmically, one performs the following steps: First compute the weights of the variables in the DDE. Determine all monomials of rank R in the components of $\mathbf{u}_n$ and their t-derivatives. Use the DDE to replace all the t-derivatives. Make a linear combination of the resulting monomials with coefficients c_i. Compute $D_t\mathbf{G}$ and remove all $\dot{\mathbf{u}}_{n-1}, \dot{\mathbf{u}}_n, \dot{\mathbf{u}}_{n+1}$, etc. Equate the resulting expression to the Fréchet derivative $\mathbf{F}'(\mathbf{u}_n)[\mathbf{G}]$ and solve the system for the c_i, treating the monomials in $\mathbf{u}_n$ and its shifts as independent. Details are in [2, 5]. For (2), the symmetry $\mathbf{G}$ of rank $R = 3$ is listed in Table 2. There are infinitely many symmetries, all with different ranks.

See [3] for the complete algorithm and its implementation in *Mathematica*, and [4] for an integrated *Mathematica* Package that computes symmetries of PDEs and DDEs.

Notes:

(i) If PDEs or DDEs are of second or higher order in t, like the Boussinesq equation in [1], we assume that they can be recast in the form given in Table 1.

(ii) A slight modification of the methods in Section 2 and 3 allows one to find invariants and symmetries that explicitly depend on t and x. See next section for an example.

(iii) Applied to systems with free parameters, the linear system for the c_i will depend on these parameters. A careful analysis of the eliminant leads to conditions on these parameters so that a sequence of invariants or symmetries exists.

(iv) For equations that lack uniformity in rank, we introduce one or more auxiliary (constant) parameters with weights. After the form of the invariant or symmetry is determined, the auxiliary parameters can be reset to one.

(v) Higher-order symmetries, such as (5) lead to new integrable evolution equations. For example, $u_t = 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}$ is the completely integrable fifth-order KdV equation due to Lax.

Details about these 5 items are given in [1, 2, 3, 5].

4 Examples

4.1 Vector Modified KdV Equation

In [7, Eq. (4)], Verheest investigates the integrability of a vector form of the modified KdV equation (vmKdV), which upon projection, reads

$$\begin{aligned} u_t + 3u^2u_x + v^2u_x + 2uvv_x + u_{3x} &= 0, \\ v_t + 3v^2v_x + u^2v_x + 2uvu_x + v_{3x} &= 0. \end{aligned} \tag{6}$$

With our software **InvariantsSymmetries.m** [4] we computed the following invariants:

$$\rho_1 = u, \quad \rho_2 = v, \quad \rho_3 = u^2 + v^2, \tag{7}$$

$$\rho_4 = \frac{1}{2}(u^2+v^2)^2 - (u_x^2 + v_x^2), \tag{8}$$

$$\rho_5 = \frac{1}{3}x(u^2+v^2) - \frac{1}{2}t(u^2+v^2)^2 + t(u_x^2+v_x^2). \tag{9}$$

Note that the latter invariant depends explicitly on x and t. Verheest [7] has shown that (6) is non-integrable for it lacks a bi-Hamiltonian structure and recursion operator. We were unable to find additional polynomial invariants. Polynomial higher-order symmetries for (6) do not appear to exist.

4.2 Extended Lotka-Volterra Equation

Itoh [6] studied the following extended version of the Lotka-Volterra equation (2),

$$\dot{u}_n = \sum_{r=1}^{k-1}(u_{n-r} - u_{n+r})u_n. \tag{10}$$

For $k = 2$, (10) is (2), for which three invariants and one symmetry are listed in Table 2. In [5], we give two additional invariants; in [3] we list two more symmetries.

For (10), we computed 5 invariants and 2 higher-order symmetries for $k = 3$ through $k = 5$.

Here is a partial list of our results:

Case 1: $k = 3$

Invariants:

$$\rho_1 = u_n, \qquad \rho_2 = \frac{1}{2}u_n^2 + u_n(u_{n+1} + u_{n+2}), \tag{11}$$

$$\begin{aligned} \rho_3 &= \frac{1}{3}u_n^3 + u_n^2(u_{n+1} + u_{n+2}) + u_n(u_{n+1} + u_{n+2})^2 \\ &\quad + u_n(u_{n+1}u_{n+3} + u_{n+2}u_{n+3} + u_{n+2}u_{n+4}). \end{aligned} \tag{12}$$

Higher-order symmetry:

$$\begin{aligned} G &= u_n^2(u_{n+1} + u_{n+2} - u_{n-2} - u_{n-1}) + u_n[(u_{n+1} + u_{n+2})^2 - (u_{n-2} + u_{n-1})^2] \\ &\quad + u_n[u_{n+1}u_{n+3} + u_{n+2}u_{n+3} + u_{n+2}u_{n+4} - (u_{n-4}u_{n-2} + u_{n-3}u_{n-2} + u_{n-3}u_{n-1})]. \end{aligned} \tag{13}$$

Case 2: $k = 4$

Invariants:

$$\rho_1 = u_n, \qquad \rho_2 = \frac{1}{2}u_n^2 + u_n(u_{n+1} + u_{n+2} + u_{n+3}), \tag{14}$$

$$\begin{aligned} \rho_3 &= \frac{1}{3}u_n^3 + u_n^2(u_{n+1} + u_{n+2} + u_{n+3}) + u_n(u_{n+1} + u_{n+2} + u_{n+3})^2 \\ &\quad + u_n(u_{n+1}u_{n+4} + u_{n+2}u_{n+4} + u_{n+3}u_{n+4} + u_{n+2}u_{n+5} + u_{n+3}u_{n+5} + u_{n+3}u_{n+6}). \end{aligned} \tag{15}$$

Higher-order symmetry:

$$\begin{aligned} G &= u_n[u_{n+1}u_{n+4} + u_{n+2}u_{n+4} + u_{n+3}u_{n+4} + u_{n+2}u_{n+5} + u_{n+3}u_{n+5} + u_{n+3}u_{n+6} \\ &\quad - (u_{n-6}u_{n-3} + u_{n-5}u_{n-3} + u_{n-4}u_{n-3} + u_{n-5}u_{n-2} - u_{n-4}u_{n-2} + u_{n-4}u_{n-1})] \\ &\quad + u_n[(u_{n+1} + u_{n+2} + u_{n+3})^2 - u_n(u_{n-3} + u_{n-2} + u_{n-1})^2] \\ &\quad + u_n^2[u_{n+1} + u_{n+2} + u_{n+3} - (u_{n-3} + u_{n-2} + u_{n-1})]. \end{aligned} \tag{16}$$

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