Computation of Higher-order Symmetries for Nonlinear Evolution and Lattice Equations*

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A straightforward algorithm for the symbolic computation of higher-order symmetries of nonlinear evolution equations and lattice equations is presented. The scaling properties of the evolution or lattice equations are used to determine the polynomial form of the higher-order symmetries. The coefficients of the symmetry can be found by solving a linear system. The method applies to polynomial systems of PDEs of first-order in time and arbitrary order in one space variable. Likewise, lattices must be of first order in time but may involve arbitrary shifts in the discretized space variable.

The algorithm is implemented in *Mathematica* and can be used to test the integrability of both nonlinear evolution equations and semi-discrete lattice equations. With our **Integrability Package**, higher-order symmetries are obtained for several well-known systems of evolution and lattice equations. For PDEs and lattices with parameters, the code allows one to determine the conditions on these parameters so that a sequence of higher-order symmetries exist. The existence of a sequence of such symmetries is a predictor for integrability.

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1 – Introduction

A large number of physically important nonlinear models are completely integrable, which means that they are linearizable via an explicit transformation or solvable via the Inverse Scattering Transform. Integrable continuous or discrete models arise in key branches of physics including classical, quantum, particle, statistical, and plasma physics. Integrable equations also model wave phenomena in nonlinear optics and the bio-sciences. Mathematically, nonlinear models involve ordinary or partial differential equations (ODEs or PDEs), differential-difference equations (DDEs), integral equations, etc. [8].

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Whichever form they come in, completely integrable equations exhibit analytic properties reflecting their rich mathematical structure. For instance, completely integrable PDEs and DDEs possess infinitely many symmetries and conserved quantities (if the model is conservative). Perhaps after a suitable change of variables, the equations have the Painlevé property, admit Bäcklund transformations, prolongation structures, or can be written in bi-Hamiltonian form [8].

The existence of an infinite hierarchy of symmetries for integrable equations can be established by explicitly constructing the recursion operator that connects such symmetries. Finding symmetries and recursion operators for nonlinear models is a nontrivial task, in particular if attempted with pen and paper. Computer algebra systems can greatly assist in the search for higher-order symmetries and recursion operators.

In this paper we present a direct algorithm that allows one to automatically compute polynomial higher-order symmetries for polynomial PDEs in 1+1 dimension and polynomial DDEs (semi-discrete equations). The systems of DDEs or PDEs must be of evolution type, i.e. first order in (continuous) time. The number of equations and the order of differentiation (or shift level) in the spatial variable are arbitrary.

We use the dilation invariance of the given system of PDEs or DDEs to determine the form of the polynomial generalized symmetry. Upon substitution of the form of the symmetry into the defining equation, one has to solve a linear system for the unknown constant coefficients of the symmetry. In case the original system contains free parameters, the eliminant of that linear system will determine the necessary conditions for the parameters, so that the system admits the required generalized symmetry. Our algorithm can thus be used as an integrability test for classes of equations involving parameters.

For the PDE case, a slight extension of our algorithm allows one to compute higher-order symmetries that *explicitly* depend on the independent variables x and t. However, in such cases, it is necessary to specify the highest degree of the independent variables in the generalized symmetry.

Once the generalized symmetries are explicitly known, it is quite often possible to find the recursion operator by inspection [7]. If the recursion operator is *hereditary* then the equation will possess infinitely many symmetries. If the operator is hereditary and *factorizable* then the equation has infinitely many conserved quantities [8, 11].

For Lagrangian systems the set of higher-order symmetries can be shown to lead to the set of conservation laws [29, 32]. For equations without Lagrangian structure there is no universal correspondence between symmetries and conservation laws. For *Mathematica* algorithms and code to compute conservations laws of nonlinear PDEs and DDEs we refer to [13] and [14, 16], respectively. The relationship between symmetries and conservation laws, as expressed through Noether's theorem, is beyond the scope of this paper (see [32] for details).

The paper is organized as follows. In Section 2 we give the definition of generalized symmetry. We also give the three steps of the algorithm to compute generalized symmetries of nonlinear evolution equations. The Korteweg-de Vries and Boussinesq equations are used to illustrate the computations. In Section 3 we extend the algorithm to nonlinear DDEs (the semi-discrete case), with the Toda lattice as leading example. The examples in Section 4 are meant to show how the technique works for complex equations, like the nonlinear Schrödinger equation and one of its discretizations (the Ablowitz-Ladik system). We use a class of fifth-order KdV-type equations, and the Hirota-Satsuma system to exemplify the method for equations with parameters. In Section 5 we give details about the use of our symmetry package in *Mathematica*. We briefly discuss other software for higher-order symmetries in Section 6. Conclusions and an outlook for future research are given in Section 7.

2 – Symmetries of Partial Differential Equations

2.1 – Definition

Consider a system of PDEs in the (single) space variable x and time variable t,

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ..., \mathbf{u}_{mx}), \tag{1}$$

where $\mathbf{u}$ and $\mathbf{F}$ are vector dynamical variables with the same number of components: $\mathbf{u} = (u_1, u_2, ..., u_n), \mathbf{F} = (F_1, F_2, ..., F_n)$ and $\mathbf{u}_{mx} = \frac{\partial^m \mathbf{u}}{\partial x^m}$. The vector function $\mathbf{F}$ is assumed to be a polynomial function in $\mathbf{u}, \mathbf{u}_x, ..., \mathbf{u}_{mx}$. There are no restrictions on the order of the system or its degree of nonlinearity. If PDEs are of second or higher order in t, we assume that they can be recast in the form (1).

A vector function $\mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$, with $\mathbf{G} = (G_1, G_2, ..., G_n)$, is called a *symmetry* of (1) if and only if it leaves (1) invariant for the replacement $\mathbf{u} \to \mathbf{u}+\epsilon\mathbf{G}$ within order ϵ. Hence,

$$\mathrm{D}_t(\mathbf{u} + \epsilon\mathbf{G}) = \mathbf{F}(\mathbf{u} + \epsilon\mathbf{G}) \tag{2}$$

must hold up to order ϵ on the solutions of (1). Consequently, $\mathbf{G}$ must satisfy the linearized equation [6, 8]

$$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}], \tag{3}$$

where $\mathbf{F}'$ is the Fréchet derivative of $\mathbf{F}$, i.e.,

$$\mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u} + \epsilon\mathbf{G})_{|\epsilon=0}. \tag{4}$$

In (2) and (4) we infer that $\mathbf{u}$ is replaced by $\mathbf{u} + \epsilon\mathbf{G}$, and $\mathbf{u}_{nx}$ by $\mathbf{u}_{nx} + \epsilon\mathrm{D}_x^n\mathbf{G}$. As usual, D_t and D_x are total derivatives.

Symmetries of the form $\mathbf{G}(x,t,\mathbf{u})$ are called *point* symmetries. If $\mathbf{G}(x,t,\mathbf{u},\mathbf{u}_x,\mathbf{u}_t)$ they are called *classical* or Lie-Bäcklund symmetries, and all others symmetries involving higher derivatives than the first are called generalized or *higher-order* symmetries [28].

The examples used in the description of the algorithm, involve one or two dependent variables. For simplicity of notation, the components of $\mathbf{u}$ will be denoted by $u, v, ...$ (instead of u_1, u_2, etc.).

2.2 – Algorithm

To illustrate our algorithm, we consider the Korteweg-de Vries (KdV) equation

$$u_t = 6uu_x + u_{3x} \tag{5}$$

from soliton theory. This ubiquitous evolution equation is known to have infinitely many symmetries [7].

Key to our method is the observation that (5) is invariant under the dilation symmetry (or scaling)

$$(t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2 u), \tag{6}$$

where λ is an arbitrary parameter. The result of this dimensional analysis can be stated as follows: u corresponds to two derivatives with respect to x, for short, $u \sim \frac{\partial^2}{\partial x^2}$. Similarly, $\frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$. Scaling invariance, which is a special Lie-point symmetry, is an intrinsic property of many integrable nonlinear PDEs and DDEs.

Our algorithm exploits this scaling invariance to find symmetries, which now proceeds in three steps.

Step 1: Determine the weights of variables

The *weight*, w, of a variable is by definition equal to the number of derivatives with respect to x the variable carries. Weights are rational, and weights of dependent variables are nonnegative. We set $w(\frac{\partial}{\partial x}) = 1$. In view of (6), we have $w(u) = 2$ and $w(\frac{\partial}{\partial t}) = 3$. Consequently, $w(x) = -1$ and $w(t) = -3$.

The *rank* of a monomial is defined as the total weight of the monomial, again in terms of derivatives with respect to x. Observe that (5) is an equation of rank 5, since all the terms (monomials) have the same rank, namely 5. This property is called *uniformity in rank.*

Conversely, requiring uniformity in rank for (5) allows one to compute the weights of the dependent variables. Indeed, with $w(\frac{\partial}{\partial x}) = 1$ we have

$$w(u) + w(\frac{\partial}{\partial t}) = 2w(u) + 1 = w(u) + 3,$$

which yields $w(u) = 2$, $w(\frac{\partial}{\partial t}) = 3$. Hence, $w(t) = -3$, which is consistent with (6).

Step 2: Construct the form of the symmetry

As an example, let us compute the form of the symmetry of rank 7. Start by listing all powers in u with rank 7 or less: $\mathcal{L}=\{1, u, u^2, u^3\}$. Next, for each monomial in $\mathcal{L}$, introduce enough x-derivatives, so that each term exactly has rank 7. Thus,

$$\frac{\partial}{\partial x}(u^3) = 3u^2 u_x, \quad \frac{\partial^3}{\partial x^3}(u^2) = 6u_x u_{2x} + 2uu_{3x}, \quad \frac{\partial^5}{\partial x^5}(u) = u_{5x}, \quad \frac{\partial^7}{\partial x^7}(1) = 0.$$

Then, gather the resulting (non-zero) terms in a set $\mathcal{R} = \{u^2 u_x, u_x u_{2x}, uu_{3x}, u_{5x}\}$, which contains the building blocks of the symmetry. Linear combination of the monomials in $\mathcal{R}$ with constant coefficients c_i gives the form of the symmetry:

$$G = c_1\, u^2 u_x + c_2\, u_x u_{2x} + c_3\, uu_{3x} + c_4\, u_{5x}. \tag{7}$$

Step 3: Determine the unknown coefficients in the symmetry

We determine the coefficients c_i by requiring that (3) holds on the solutions of (1). Compute $\mathrm{D}_t\mathbf{G}$ and use (1) to remove $\mathbf{u}_t, \mathbf{u}_{tx}, \mathbf{u}_{txx}$, etc. For given $\mathbf{F}$, compute the Fréchet derivative (4) and, in view of (3), equate the resulting expressions. Treating the different monomial terms in $\mathbf{u}$ and its x-derivatives as independent, the linear system for the coefficients c_i is readily obtained.

For (5), we perform this computation with $F = 6uu_x + u_{3x}$ and G in (7). Considering as independent all products and powers of $u, u_x, u_{xx}, \ldots$, in

$$\begin{aligned}&(12c_1 - 18c_2)u_x^2 u_{2x} + (6c_1 - 18c_3)uu_{2x}^2 + (6c_1 - 18c_3)uu_x u_{3x} + (3c_2 - 60c_4)u_{3x}^2 + \\ &(3c_2 + 3c_3 - 90c_4)u_{2x}u_{4x} + (3c_3 - 30c_4)u_x u_{5x} \equiv 0,\end{aligned} \tag{8}$$

we obtain the linear system for the coefficients c_i :

$\mathcal{S} = \{12c_1 – 18c_2 = 0, 6c_1 – 18c_3 = 0, 3c_2 – 60c_4 = 0, 3c_2 + 3c_3 – 90c_4 = 0, 3c_3 – 30c_4 = 0\}$.

The solution is $\frac{c_1}{30} = \frac{c_2}{20} = \frac{c_3}{10} = c_4$. Since symmetries can only be determined up to a multiplicative constant, we choose $c_1 = 30, c_2 = 20, c_3 = 10$ and $c_4 = 1$, and substitute this into (7). Hence,

$$G = 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}.$$

Note that $u_t = G$ is known as the Lax equation, which is the fifth-order PDE in the completely integrable KdV hierarchy.

Analogously, for (5) we computed the (x–t independent) symmetries of rank ≤ 11. They are:

$$\begin{aligned}G^{(1)} &= u_x, \quad G^{(2)} = 6uu_x + u_{3x}, \quad G^{(3)} = 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}, \\ G^{(4)} &= 140u^3 u_x + 70u_x^3 + 280uu_x u_{2x} + 70u^2 u_{3x} + 70u_{2x}u_{3x} + 42u_x u_{4x} \\ &\quad + 14uu_{5x} + u_{7x},\end{aligned}$$

$$
\begin{aligned}
G^{(5)} = \; & 630u^4u_x + 1260uu_x^3 + 2520u^2u_xu_{2x} + 1302u_xu_{2x}^2 + 420u^3u_{3x} \\
& +966u_x^2u_{3x} + 1260uu_{2x}u_{3x} + 756uu_xu_{4x} + 252u_{3x}u_{4x} \\
& +126u^2u_{5x} + 168u_{2x}u_{5x} + 72u_xu_{6x} + 18uu_{7x} + u_{9x}.
\end{aligned}
$$

These results agree with those listed in the literature (see e.g. [7, 28, 30]).

Remarks.

(i) The recursion operator [30, p. 312] for the KdV equation is given by

$$\mathcal{R} = D^2 + 4u + 2u_xD^{-1}. \tag{9}$$

This operator is hereditary [11] and connects the above symmetries.

For example,

$$
\begin{aligned}
\mathcal{R}u_x &= (D^2 + 4u + 2u_xD^{-1})u_x = 6uu_x + u_{3x}, \\
\mathcal{R}(6uu_x + u_{3x}) &= (D^2 + 4u + 2u_xD^{-1})(6uu_x + u_{3x}) \\
&= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x},
\end{aligned} \tag{10}
$$

and so forth.

(ii) Instead of working with the definition (3) of the symmetry, one could introduce an evolution equation,

$$\mathbf{u}_\tau = \mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...), \tag{11}$$

which defines the flow generated by $\mathbf{G}$ and parameterized by the auxiliary time variable τ. The symmetry can then be computed from the compatibility condition of (1) and (11):

$$\mathrm{D}_\tau\mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ..., \mathbf{u}_{nx}) = \mathrm{D}_t\mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...). \tag{12}$$

One then proceeds as follows: As above, determine the form of the symmetry $\mathbf{G}$ involving the constant coefficients c_i. Then, compute $\mathrm{D}_t\mathbf{G}$ and use (1) to remove $\mathbf{u}_t, \mathbf{u}_{tx}$, etc. Subsequently, compute $\mathrm{D}_\tau\mathbf{F}$ and use (11) to remove $\mathbf{u}_\tau, \mathbf{u}_{\tau x}$, etc. Finally, use (12) to determine the linear system for the unknown c_i. Solve the system and substitute the result into the form of $\mathbf{G}$.

Applied to our example, D_tG is computed with G in (7). Next, (5) is used to eliminate all t-derivatives of u from the expression of D_tG. Then, compute $\mathrm{D}_\tau F$ with F in the right hand side of (5), and eliminate all τ-derivatives through (11) after substitution of (7). Finally, expressing that $\mathrm{D}_\tau F - D_tG \equiv 0$ leads to (8).

Although this procedure (see Ito [23]) circumvents the evaluation of the Fréchet derivative, it seems more involved than our algorithm which uses the definition (3).

2.3 – Symmetries Explicitly Dependent on x and t

The KdV equation (5) has also symmetries which *explicitly* depend on x and t. Our algorithm can be used to find these symmetries provided that we specify the maximum degree in x and t.

As an example, we will compute the symmetry of rank 2 for (5), that linearly depends on x and/or t. In other words, the highest degree in x or t in the symmetry is 1.

We start with the list of monomials in u, x and t of rank 2 or less:

$$\mathcal{L} = \{1, u, x, xu, t, tu, tu^2\}.$$

Then, for each monomial in $\mathcal{L}$, introduce enough x-derivatives, so that each term exactly has weight 2. Thus,

$$\mathrm{D}_x(xu) = u + xu_x,\ \mathrm{D}_x(tu^2) = 2tuu_x,\ \mathrm{D}_x^3(tu) = tu_{3x},\ \mathrm{D}_x^2(1) = \mathrm{D}_x^3(x) = \mathrm{D}_x^5(t) = 0.$$

Gather the non-zero resulting terms in a set $\mathcal{R} = \{u, xu_x, tuu_x, tu_{3x}\}$, which contains the building blocks of the symmetry. Linear combination of the monomials in $\mathcal{R}$ with constant coefficients c_i gives the form of the symmetry:

$$G = c_1\, u + c_2\, xu_x + c_3\, tuu_x + c_4\, tu_{3x}. \tag{13}$$

Now, determine the coefficients c_1 through c_4 by requiring that (3) holds on the solutions of (5). After grouping the terms, one gets

$$(6c_1 + 6c_2 - c_3)uu_x + (3c_3 - 18c_4)tu_{2x}^2 + (3c_2 - c_4)u_{3x} + (3c_3 - 18c_4)tu_x u_{3x} \equiv 0,$$

which yields

$$\mathcal{S} = \{6c_1 + 6c_2 - c_3 = 0, 3c_3 - 18c_4 = 0, 3c_2 - c_4 = 0\}.$$

The solution is $\frac{3c_1}{2} = 3c_2 = \frac{c_3}{6} = c_4$. We choose $c_1 = \frac{2}{3}, c_2 = \frac{1}{3}, c_3 = 6$ and $c_4 = 1$, and substitute this into (13). Hence,

$$G = \frac{2}{3}u + \frac{1}{3}xu_x + 6tuu_x + tu_{3x}.$$

Similarly, we computed other symmetries of (5) that linearly depend on x and t. They are of rank 0 and 2 :

$$G = 1 + 6tu_x, \quad \text{and} \quad G = 2u + xu_x + tu_t = 2u + xu_x + 6tuu_x + tu_{3x}.$$

Our results agree with those in the literature [28].

2.4 – Example: Boussinesq equation

For scaling invariant systems such as (5), it suffices to consider the dilation symmetry on the space of independent and dependent variables. For systems that are inhomogeneous under a suitable scaling symmetry, such as the example given below, we use the following trick: We introduce one (or more) auxiliary parameter(s) with an appropriate scaling. These extra parameters can be viewed as additional dependent variables, however, their derivatives are zero. By extending the action of the dilation symmetry to the space of independent and dependent variables, *including* the parameters, we are able to apply our algorithm to a larger range of polynomial PDE systems.

Consider the wave equation,

$$u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0, \tag{14}$$

(α constant) which was proposed by Boussinesq to describe surface water waves whose horizontal scale is much larger than the depth of the water [1].

To apply our algorithm, we must first rewrite (14) as a first-order system,

$$\begin{aligned} u_t &= v_x, \\ v_t &= u_x - 3uu_x - \alpha u_{3x}, \end{aligned} \tag{15}$$

where v is an auxiliary dependent variable. It is easy to verify that the terms u_x and αu_{3x} in the second equation obstruct uniformity in rank. To circumvent the problem we introduce an auxiliary parameter β with (unknown) weight, and replace (15) by

$$\begin{aligned} u_t &= v_x, \\ v_t &= \beta u_x - 3uu_x - \alpha u_{3x}. \end{aligned} \tag{16}$$

As described in Step 1 we compute the weights from

$$\begin{aligned} & w(u) + w(\frac{\partial}{\partial t}) = w(v) + 1, \\ & w(v) + w(\frac{\partial}{\partial t}) = w(\beta) + w(u) + 1 = 2w(u) + 1 = w(u) + 3. \end{aligned}$$

This yields

$$w(u) = 2,\ w(v) = 3,\ w(\beta) = 2,\ \text{ and } w(\frac{\partial}{\partial t}) = -w(t) = 2,$$

and the scaling properties of (16) are $u \sim \beta \sim \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2}$, $v \sim \frac{\partial^3}{\partial x^3}$. Indeed, (16) is invariant under the dilation symmetry

$$(x, t, u, v, \beta) \rightarrow (\lambda^{-1}x, \lambda^{-2}t, \lambda^2 u, \lambda^3 v, \lambda^2 \beta).$$

Observe that all the monomials in the equations in (16) have rank 4 and 5. Therefore, for any symmetry $\mathbf{G}$ of (16),

$$\mathrm{rank}(G_2) = \mathrm{rank}(G_1) + 1 = \mathrm{rank}(G_1) + w(v) - w(u).$$

Let us construct the form of the symmetry $\mathbf{G} = (G_1, G_2)$ with $\mathrm{rank}(G_1) = 6$ and $\mathrm{rank}(G_2) = 7$. First, list all monomials in u, v and β of rank 6 (respectively rank 7) or less:

$$\begin{aligned}
\mathcal{L}_1 &= \{1, \beta, \beta^2, \beta^3, u, \beta u, \beta^2 u, u^2, \beta u^2, u^3, v, \beta v, uv, v^2\}, \\
\mathcal{L}_2 &= \{1, \beta, \beta^2, \beta^3, u, \beta u, \beta^2 u, u^2, \beta u^2, u^3, v, \beta v, \beta^2 v, uv, \beta uv, u^2 v, v^2\}.
\end{aligned}$$

Next, for each monomial in $\mathcal{L}_1$ and $\mathcal{L}_2$, introduce the necessary x-derivatives, so that each term in $\mathcal{L}_1$ exactly has rank 6, and each term in $\mathcal{L}_2$ has rank 7. Keeping in mind that β is constant, and proceeding with the rest of the algorithm, we obtain:

$$\begin{aligned}
G_1^{(1)} &= u_x v + u v_x + \frac{2}{3}\alpha v_{3x}, \\
G_2^{(1)} &= \beta u u_x - 3u^2 u_x + v v_x - 6\alpha u_x u_{2x} + \frac{2}{3}\alpha\beta u_{3x} - 3\alpha u u_{3x} - \frac{2}{3}\alpha^2 u_{5x}.
\end{aligned} \tag{17}$$

Finally, setting $\beta = 1$ in (17), one obtains a symmetry of (15) although initially this system was not uniform in rank. We list one more higher-order symmetry of (15):

$$\begin{aligned}
G_1^{(2)} &= u u_x - \frac{3}{2}u^2 u_x + v v_x - 5\alpha u_x u_{2x} + \frac{2}{3}\alpha u_{3x} - 2\alpha u u_{3x} - \frac{8}{15}\alpha^2 u_{5x}, \\
G_2^{(2)} &= u v_x + v u_x - 3u u_x v - \frac{3}{2}u^2 v_x - 2\alpha u_{2x} v_x - 3\alpha u_x v_{2x} - \alpha u_{3x} v + \frac{2}{3}\alpha v_{3x} \\
&\quad -2\alpha u v_{3x} - \frac{8}{15}\alpha^2 v_{5x}.
\end{aligned}$$

3 – Symmetries of Differential-difference Equations

3.1 – Definition

Consider a system of DDEs,

$$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...), \tag{18}$$

where the equations are continuous in time, and discretized in the (single) space variable. As before, $\mathbf{u}_n$ and $\mathbf{F}$ are vector dynamical variables with any number of components, and $\mathbf{F}$ is assumed to be a polynomial with constant coefficients.

There are no restrictions on the level of shifts or the degree of nonlinearity. If DDEs are of second or higher order in t, they must be recast in the form (18).

A vector function $\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ is called a *symmetry* of (1) if the infinitesimal transformation

$$\mathbf{u}_n \rightarrow \mathbf{u}_n + \epsilon \mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...) \tag{19}$$

leaves (18) invariant within order ϵ.

Consequently, $\mathbf{G}$ must satisfy the linearized equation [6, 8]

$$\mathrm{D}_t \mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}], \tag{20}$$

where $\mathbf{F}'$ is the Fréchet derivative of $\mathbf{F}$, defined as

$$\mathbf{F}'(\mathbf{u}_n)[\mathbf{G}] = \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u}_n + \epsilon \mathbf{G})_{|\epsilon=0}. \tag{21}$$

Of course, (19) means that $\mathbf{u}_{n+k}$ is replaced by $\mathbf{u}_{n+k} + \epsilon \mathbf{G}_{|n \rightarrow n+k}$. For compactness of notation, in (20) and (21) we used $\mathbf{F}'(\mathbf{u}_n)$ instead of $\mathbf{F}'(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$.

Also for notational simplicity, in the description of the algorithm below, the components of $\mathbf{u}_n$ will be denoted by u_n, v_n, etc. We use $F_1, F_2, ...$ and $G_1, G_2, ...$, to denote the components of $\mathbf{F}$ and $\mathbf{G}$, respectively.

3.2 – Algorithm

As the leading example, we consider the one-dimensional lattice [17, 38]

$$\ddot{y}_n = \exp{(y_{n-1} - y_n)} - \exp{(y_n - y_{n+1})}, \tag{22}$$

due to Toda. In (22), y_n is the displacement from equilibrium of the nth particle with unit mass under an exponential decaying interaction force between nearest neighbors. With the change of variables,

$$u_n = \dot{y}_n, \qquad v_n = \exp{(y_n - y_{n+1})},$$

the Toda lattice (22) can be written in polynomial form

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}). \tag{23}$$

Observe that (23) is invariant under the dilation symmetry

$$(t, u_n, v_n) \rightarrow (\lambda^{-1} t, \lambda u_n, \lambda^2 v_n), \tag{24}$$

where λ is an arbitrary parameter. Thus, u_n corresponds to one derivative with respect to t, or $u_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$, and, similarly, $v_n \sim \frac{\mathrm{d}^2}{\mathrm{dt}^2}$. Our 3-step algorithm exploits this scaling property to find symmetries.

Step 1: Determine the weights of variables

In contrast to the algorithm for PDEs, we have to define the *weight*, w, of variables in terms of the number of derivatives with respect to t, and we set $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$. Weights of dependent variables are nonnegative, rational, and independent of n. In view of (24), we have $w(u_n) = 1$, and $w(v_n) = 2$.

The *rank* of a monomial is defined as the total weight of the monomial, again in terms of derivatives with respect to t. Observe that in the first equation of (23), all the monomials have the same rank, namely 2, and in the second equation, all the terms have rank 3.

Conversely, requiring uniformity in rank for each equation in (23) allows one to compute the weights of the dependent variables. Indeed,

$$w(u_n) + 1 = w(v_n), \quad w(v_n) + 1 = w(u_n) + w(v_n),$$

yields $w(u_n) = 1$, $w(v_n) = 2$, which is consistent with (24).

Step 2: Construct the form of the symmetry

As an example, we compute the form of the symmetry of rank $(3, 4)$, i.e. G_1 and G_2 will have ranks 3 and 4, respectively. Start by listing all monomials in u_n and v_n of ranks 3 and 4, or less:

$$\begin{aligned}\mathcal{L}_1 &= \{u_n^3, u_n^2, u_n v_n, u_n, v_n\},\\ \mathcal{L}_2 &= \{u_n^4, u_n^3, u_n^2 v_n, u_n^2, u_n v_n, u_n, v_n^2, v_n\}.\end{aligned}$$

Next, for each monomial in $\mathcal{L}_1$ and $\mathcal{L}_2$, introduce the necessary t-derivatives. so that each term exactly has rank 3 and 4, respectively. At the same time, use (23) to remove all $t-$derivatives. Doing so, based on $\mathcal{L}_1$, we obtain

$$\frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n^3) = u_n^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,$$

$$\frac{\mathrm{d}}{\mathrm{dt}}(u_n^2) = 2u_n \dot{u}_n = 2u_n v_{n-1} - 2u_n v_n, \qquad \frac{\mathrm{d}}{\mathrm{dt}}(v_n) = \dot{v}_n = u_n v_n - u_{n+1} v_n,$$

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = \frac{\mathrm{d}}{\mathrm{dt}}(\dot{u}_n) = \frac{\mathrm{d}}{\mathrm{dt}}(v_{n-1} - v_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n.$$

Gather the resulting terms in a set: $\mathcal{R}_1 = \{u_n^3, u_{n-1} v_{n-1}, u_n v_{n-1}, u_n v_n, u_{n+1} v_n\}$. Similarly, based on the monomials in $\mathcal{L}_2$, we get

$$\begin{aligned}\mathcal{R}_2 = \{&u_n^4, u_{n-1}^2 v_{n-1}, u_{n-1} u_n v_{n-1}, u_n^2 v_{n-1}, v_{n-2} v_{n-1}, v_{n-1}^2, u_n^2 v_n,\\ &u_n u_{n+1} v_n, u_{n+1}^2 v_n, v_{n-1} v_n, v_n^2, v_n v_{n+1}\}.\end{aligned}$$

Linear combination of the monomials in $\mathcal{R}_1$ and $\mathcal{R}_2$ with constant coefficients c_i gives the explicit form of the symmetry:

$$\begin{aligned}
G_1 &= c_1\,u_n^3 + c_2\,u_{n-1}v_{n-1} + c_3\,u_n v_{n-1} + c_4\,u_n v_n + c_5\,u_{n+1}v_n, \\
G_2 &= c_6\,u_n^4 + c_7\,u_{n-1}^2 v_{n-1} + c_8\,u_{n-1}u_n v_{n-1} + c_9\,u_n^2 v_{n-1} + c_{10}\,v_{n-2}v_{n-1} + \\
&\quad c_{11}\,v_{n-1}^2 + c_{12}\,u_n^2 v_n + c_{13}\,u_n u_{n+1} v_n + c_{14}\,u_{n+1}^2 v_n + c_{15}\,v_{n-1}v_n + \\
&\quad c_{16}\,v_n^2 + c_{17}\,v_n v_{n+1}.
\end{aligned} \tag{25}$$

Step 3: Determine the unknown coefficients in the symmetry

To determine the coefficients c_i we require that (20) holds on any solution of (18). Compute $\mathrm{D}_t\mathbf{G}$ and use (18) to remove all $\dot{\mathbf{u}}_{n-1}, \dot{\mathbf{u}}_n, \dot{\mathbf{u}}_{n+1}$, etc. Compute the Fréchet derivative (21) and, in view of (20), equate the resulting expressions. Considering as independent all the monomials in $\mathbf{u}_n$ and their shifts, we obtain the linear system that determines the coefficients c_i.

Applied to (23) with (25), we obtain the solution

$$c_1 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0, \tag{26}$$

$$-c_2 = -c_3 = c_4 = c_5 = -c_{12} = c_{14} = -c_{15} = c_{17}. \tag{27}$$

Therefore, with the choice $c_{17} = 1$, the symmetry is

$$\begin{aligned}
G_1 &= u_n v_n - u_{n-1}v_{n-1} + u_{n+1}v_n - u_n v_{n-1}, \\
G_2 &= u_{n+1}^2 v_n - u_n^2 v_n + v_n v_{n+1} - v_{n-1}v_n.
\end{aligned} \tag{28}$$

It is easy to produce new completely integrable DDEs based on these symmetries. For instance, the DDE system

$$\begin{aligned}
\dot{u}_n &= G_1 = u_n v_n - u_{n-1}v_{n-1} + u_{n+1}v_n - u_n v_{n-1}, \\
\dot{v}_n &= G_2 = u_{n+1}^2 v_n - u_n^2 v_n + v_n v_{n+1} - v_{n-1}v_n.
\end{aligned} \tag{29}$$

is also completely integrable.

To illustrate the effectiveness of our algorithm to filter out integrable cases among systems of DDEs with parameters, consider a parameterized version of the Toda lattice,

$$\dot{u}_n = \alpha\, v_{n-1} - v_n, \quad \dot{v}_n = v_n\,(\beta\, u_n - u_{n+1}), \tag{30}$$

where α and β are *nonzero* constants. In [31] it was shown that (30) is completely integrable if and only if $\alpha = \beta = 1$.

Using our algorithm, one can easily compute the *compatibility conditions* for α and β, so that (30) admits a polynomial symmetry, say, of rank $(3, 4)$. The steps are the same as for (23). However, the linear system for the c_i is parameterized by α and β and must be analyzed carefully. This analysis leads to the condition $\alpha = \beta = 1$.

For $\alpha = \beta = 1$, (30) coincides with (23), for which we computed symmetries with ranks $(4, 5)$ and $(5, 6)$. They are:

$$\begin{aligned}
G_1^{(1)} &= u_n^2 v_n + u_n u_{n+1} v_n + u_{n+1}^2 v_n + v_n^2 + v_n v_{n+1} - u_{n-1}^2 v_{n-1} - u_{n-1} u_n v_{n-1} \\
&\quad - u_n^2 v_{n-1} - v_{n-2} v_{n-1} - v_{n-1}^2, \\
G_2^{(1)} &= u_{n+1} v_n^2 + 2u_{n+1} v_n v_{n+1} + u_{n+2} v_n v_{n+1} - u_n^3 v_n + u_{n+1}^3 v_n \\
&\quad - u_{n-1} v_{n-1} v_n - 2u_n v_{n-1} v_n - u_n v_n^2, \\
G_1^{(2)} &= u_n^3 v_n + u_n^2 u_{n+1} v_n + u_n u_{n+1}^2 v_n + u_{n+1}^3 v_n + 2u_n v_n^2 + 2u_{n+1} v_n^2 \\
&\quad + u_n v_n v_{n+1} + 2u_{n+1} v_n v_{n+1} + u_{n+2} v_n v_{n+1} - u_{n-1}^3 v_{n-1} - u_{n-1}^2 u_n v_{n-1} \\
&\quad - u_{n-1} u_n^2 v_{n-1} - u_n^3 v_{n-1} - u_{n-2} v_{n-2} v_{n-1} - 2u_{n-1} v_{n-2} v_{n-1} \\
&\quad - u_n v_{n-2} v_{n-1} - 2u_{n-1} v_{n-1}^2 - 2u_n v_{n-1}^2, \\
G_2^{(2)} &= u_{n+1}^4 v_n - u_n^4 v_n - u_{n-1}^2 v_{n-1} v_n - 2u_{n-1} u_n v_{n-1} v_n - 3u_n^2 v_{n-1} v_n \\
&\quad - v_{n-2} v_{n-1} v_n - v_{n-1}^2 v_n - 2u_n^2 v_n^2 + 2u_{n+1}^2 v_n^2 - v_{n-1} v_n^2 + 3u_{n+1}^2 v_n v_{n+1} \\
&\quad + 2u_{n+1} u_{n+2} v_n v_{n+1} + u_{n+2}^2 v_n v_{n+1} + v_n^2 v_{n+1} + v_n v_{n+1}^2 + v_n v_{n+1} v_{n+2}.
\end{aligned}$$

4 – More Examples

4.1 – Nonlinear Schrödinger Equation

The nonlinear Schrödinger (NLS) equation [1],

$$iq_t - q_{2x} + 2|q|^2 q = 0, \tag{31}$$

arises as an asymptotic limit of a slowly varying dispersive wave envelope in a nonlinear medium, and as such has significant applications in nonlinear optics, water waves, and plasma physics. Together with the ubiquitous KdV equation (5), the completely integrable NLS equation is one of the most studied soliton equations.

In order to compute the symmetries of (31) we consider q and q^* as independent variables and add the complex conjugate equation to (31). Absorbing i in the scale of t, we get

$$q_t - q_{2x} + 2q^2 q^* = 0, \quad q_t^* + q_{2x}^* - 2q^{*2} q = 0. \tag{32}$$

Since $w(q) = w(q^*)$, we obtain

$$w(q) = w(q^*) = 1, \quad \text{and} \quad w(\frac{\partial}{\partial t}) = -w(t) = 2.$$

Hence, the following are the symmetries of ranks $(4, 4), (5, 5)$, and $(6, 6)$:

$$G_1^{(1)} = -6qq_x q^* + q_{3x}, \qquad G_2^{(1)} = G_1^{*(1)} = -6qq^* q_x^* + q_{3x}^*,$$

$$
\begin{aligned}
G_1^{(2)} &= -6q^3 q^{*2} + 6q_x^2 q^* + 4qq_x q_x^* + 8qq_{2x}q^* + 2q^2 q_{2x}^* - q_{4x}, \quad G_2^{(2)} = G_1^{*(2)} \\
G_1^{(3)} &= 30q^2 q_x q^{*2} - 10{q_x}^2 q_x^* - 20q_x q_{2x} q^* - 10qq_{2x}q_x^* - 10qq_x q_{2x}^* \\
&\quad -10qq_{3x}q^* + q_{5x}, \qquad G_2^{(3)} = G_1^{*(3)}.
\end{aligned}
$$

4.2 – Fifth-Order Korteweg-de Vries Equations

Consider the parameterized family of fifth-order equations,

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0, \tag{33}$$

where α, β, γ are nonzero constants. Integrable cases of (33) are well known in the literature [9, 20, 26, 34]. Indeed, for $\alpha = 30, \beta = 20, \gamma = 10$, equation (33) reduces to the Lax equation [27]. The SK equation, due to Sawada and Kotera [35], and Dodd and Gibbon [5], is obtained for $\alpha = 5, \beta = 5, \gamma = 5$. The KK equation, due to Kaup [25] and Kupershmidt, corresponds to $\alpha = 20, \beta = 25, \gamma = 10$.

The scaling properties of (33) are such that $u \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^5}{\partial x^5}$. Using our algorithm, one easily computes the *compatibility conditions* for the parameters α, β and γ, so that (33) admits a symmetry of fixed rank. The results are:

Rank 3: $G = u_x$ is a symmetry of (33) without any conditions on the parameters.

Rank 5: $G = uu_x + \frac{5}{3\gamma} u_{3x}$ is a symmetry of (33) provided that

$$\alpha = \frac{3}{10}\gamma^2, \text{ and } \beta = 2\gamma. \tag{34}$$

The Lax equation satisfies (34). Since the KdV equation (5) is a member of Lax hierarchy, condition (34) comes as no surprise.

Rank 7: Equation (33) is of rank 7. The stationary part of (33) is the symmetry.

Rank 9: Three branches emerge:

(i) If condition (34) holds then

$$
\begin{aligned}
G &= u^3 u_x + \frac{5}{\gamma} u_x^3 + \frac{20}{\gamma} u u_x u_{2x} + \frac{5}{\gamma} u^2 u_{3x} + \frac{50}{\gamma^2} u_{2x} u_{3x} + \frac{30}{\gamma^2} u_x u_{4x} \\
&\quad + \frac{10}{\gamma^2} u u_{5x} + \frac{50}{7\gamma^3} u_{7x}.
\end{aligned}
$$

(ii) If

$$\alpha = \frac{1}{5}\gamma^2, \text{ and } \beta = \gamma \tag{35}$$

holds, one has the symmetry

$$
\begin{aligned}
G &= u^3 u_x + \frac{15}{4\gamma} u_x^3 + \frac{45}{2\gamma} u u_x u_{2x} + \frac{15}{2\gamma} u^2 u_{3x} + \frac{225}{4\gamma^2} u_{2x} u_{3x} + \frac{75}{2\gamma^2} u_x u_{4x} \\
&\quad + \frac{75}{4\gamma^2} u u_{5x} + \frac{375}{28\gamma^3} u_{7x}.
\end{aligned}
$$

The SK equation satisfies the condition (35).

(iii) One has the symmetry

$$\begin{aligned} G &= u^3u_x + \frac{75}{8\gamma}u_x^3 + \frac{135}{4\gamma}uu_xu_{2x} + \frac{15}{2\gamma}u^2u_{3x} + \frac{225}{2\gamma^2}u_{2x}u_{3x} + \frac{525}{8\gamma^2}u_xu_{4x} \\ &\quad + \frac{75}{4\gamma^2}uu_{5x} + \frac{375}{28\gamma^3}u_{7x} \end{aligned}$$

provided that

$$\alpha = \frac{1}{5}\gamma^2, \text{ and } \beta = \frac{5}{2}\gamma, \tag{36}$$

which holds for the KK case.

Rank 11: One obtains the symmetry

$$\begin{aligned} G &= u^4u_x + \frac{20}{\gamma}uu_x^3 + \frac{40}{\gamma}u^2u_xu_{2x} + \frac{620}{3\gamma^2}u_xu_{2x}^2 + \frac{20}{3\gamma}u^3u_{3x} + \frac{460}{3\gamma^2}u_x^2u_{3x} \\ &\quad + \frac{200}{\gamma^2}uu_{2x}u_{3x} + \frac{120}{\gamma^2}uu_xu_{4x} + \frac{400}{\gamma^3}u_{3x}u_{4x} + \frac{20}{\gamma^2}u^2u_{5x} + \frac{800}{3\gamma^3}u_{2x}u_{5x} \\ &\quad + \frac{800}{7\gamma^3}u_xu_{6x} + \frac{200}{7\gamma^3}uu_{7x} + \frac{1000}{63\gamma^4}u_{9x} \end{aligned}$$

provided that the condition (34) for the Lax hierarchy is satisfied.

In summary, our algorithm allows one to filter out all the integrable cases in the class (33). Alternatively, in [13] we investigated the conditions on the parameters α, β, γ such that (33) admits an infinite sequence (perhaps with gaps) of polynomial conservation laws. The conditions in [13] are exactly the same as the ones above.

4.3 – Hirota and Satsuma System

Hirota and Satsuma [21] proposed a coupled system of KdV equations,

$$\begin{aligned} u_t &= 6\alpha uu_x + 6vv_x + \alpha u_{3x}, \\ v_t &= 3uv_x + v_{3x}, \end{aligned} \tag{37}$$

where α is a nonzero parameter. System (37) describes the interaction of two long waves with different dispersion relations. It is known to be completely integrable provided $\alpha = -\frac{1}{2}$. The scaling properties of (37) are such that $u \sim v \sim \frac{\partial^2}{\partial x^2}$, $\frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$. System (37) is of rank 5. In a search for the symmetry of higher ranks, we obtained the symmetry of rank 7:

$$\begin{aligned} G_1 &= u^2u_x - \frac{2}{3}u_xv^2 - \frac{4}{3}uvv_x + \frac{2}{3}u_xu_{2x} - \frac{2}{3}v_xv_{2x} + \frac{1}{3}uu_{3x} - \frac{2}{3}vv_{3x} + \frac{1}{30}u_{5x}, \\ G_2 &= -\frac{1}{3}u^2v_x - \frac{2}{3}v^2v_x - \frac{1}{3}u_{2x}v_x - \frac{2}{3}u_xv_{2x} - \frac{2}{3}uv_{3x} - \frac{2}{15}v_{5x}, \end{aligned}$$

provided that $\alpha = -\frac{1}{2}$, which is the condition for complete integrability of (37).

4.4 – Volterra Chain

Consider the following integrable discretization of the KdV equation:

$$\dot{u}_n = u_n\,(u_{n+1} - u_{n-1}), \tag{38}$$

which is also known as the Kac-Van Moerbeke equation. It arises in the study of Langmuir oscillations in plasmas and in population dynamics [1, 24].

Note that (38) is invariant under the dilation symmetry $(t, u_n) \to (\lambda^{-1}t, \lambda u_n)$. Hence, u_n corresponds to one derivative with respect to t, i.e. $u_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$.

We computed the symmetries of (38) with ranks 3 through 5. They are:

$$\begin{aligned}
G^{(1)} &= u_n u_{n+1}(u_n + u_{n+1} + u_{n+2}) - u_{n-1}u_n(u_{n-2} + u_{n-1} + u_n),\\
G^{(2)} &= u_n^3 u_{n+1} + 2u_n^2 u_{n+1}^2 + u_n u_{n+1}^3 + u_n^2 u_{n+1}u_{n+2} + 2u_n u_{n+1}^2 u_{n+2}\\
&\quad + u_n u_{n+1} u_{n+2}^2 + u_n u_{n+1} u_{n+2} u_{n+3} - u_{n-3}u_{n-2}u_{n-1}u_n - u_{n-2}^2 u_{n-1} u_n\\
&\quad - 2u_{n-2}u_{n-1}^2 u_n - u_{n-1}^3 u_n - u_{n-2}u_{n-1}u_n^2 - u_{n-1}u_n^3,\\
G^{(3)} &= u_n^4 u_{n+1} + u_{n-1}u_n^2 u_{n+1}^2 + 3u_n^3 u_{n+1}^2 + 3u_n^2 u_{n+1}^3 + u_n u_{n+1}^4 + u_n^3 u_{n+1} u_{n+2}\\
&\quad + 4u_n^2 u_{n+1}^2 u_{n+2} + 3u_n u_{n+1}^3 u_{n+2} + u_n^2 u_{n+1} u_{n+2}^2 + 3u_n u_{n+1}^2 u_{n+2}^2\\
&\quad + u_n u_{n+1} u_{n+2}^3 + u_n^2 u_{n+1} u_{n+2} u_{n+3} + 2u_n u_{n+1}^2 u_{n+2} u_{n+3}\\
&\quad + u_n u_{n+1} u_{n+2} u_{n+3}^2 + u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} - u_{n-4}u_{n-3}u_{n-2}u_{n-1}u_n\\
&\quad - u_{n-3}^2 u_{n-2} u_{n-1} u_n - 2u_{n-3}u_{n-2}^2 u_{n-1} u_n - u_{n-2}^3 u_{n-1} u_n\\
&\quad - 2u_{n-3}u_{n-2}u_{n-1}^2 u_n - 3u_{n-2}^2 u_{n-1}^2 u_n - 3u_{n-2}u_{n-1}^3 u_n - u_{n-1}^4 u_n\\
&\quad - u_{n-3}u_{n-2}u_{n-1}u_n^2 - u_{n-2}u_{n-1}^2 u_n^2 - 4u_{n-2}u_{n-1}^2 u_n^2 - 3u_{n-1}^3 u_n^2\\
&\quad - u_{n-2}u_{n-1}u_n^3 - 3u_{n-1}^2 u_n^3 - u_{n-1}u_n^4 - u_{n-1}^2 u_n^2 u_{n+1}.
\end{aligned}$$

Ignoring a trivial misprint in [28], Mikhailov *et al.* listed the symmetry $G^{(1)}$.

4.5 – Ablowitz-Ladik Discretization of the Nonlinear Schrödinger Equation

In [2, 3], Ablowitz and Ladik studied some of the properties of the following integrable discretization of the NLS equation:

$$i\,\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} \pm u_n^* u_n(u_{n+1} + u_{n-1}), \tag{39}$$

where u_n^* is the complex conjugate of u_n. We continue with the plus sign; the other case would be treated analogously. Instead of splitting u_n into its real and imaginary parts, we treat u_n and $v_n = u_n^*$ as independent variables and augment (39) with its complex conjugate equation. Absorbing i in the scale on t, we get

$$\begin{aligned}
\dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n(u_{n+1} + u_{n-1}),\\
\dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n(v_{n+1} + v_{n-1}).
\end{aligned} \tag{40}$$

Since $v_n = u_n^*$, we have $w(v_n) = w(u_n)$. Neither of the equations in (40) is uniform in rank. To circumvent this problem we introduce an auxiliary parameter α with weight, and replace (40) by

$$\begin{aligned} \dot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}), \\ \dot{v}_n &= -\alpha(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned} \tag{41}$$

Uniformity in rank requires that

$$\begin{aligned} w(u_n) + 1 &= w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n), \\ w(v_n) + 1 &= w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n), \end{aligned}$$

which yields $w(u_n) = w(v_n) = \frac{1}{2}, w(\alpha) = 1$, or, $u_n^2 \sim v_n^2 \sim \alpha \sim \frac{\mathrm{d}}{\mathrm{dt}}$.

Recall that the 'uniformity in rank' requirement is essential for the first two steps of the algorithm. However, after step 2, we may set $\alpha = 1$. The computations now proceed as in the previous examples. We searched for symmetries of (40) of ranks $(2,2)$ through $(7/2, 7/2)$, and found symmetries of ranks $(5/2, 5/2)$ and $(7/2, 7/2)$. To save space, we only list the symmetries of rank $(5/2, 5/2)$:

$$\begin{aligned} G_1^{(1)} &= -u_{n+2} - u_n u_{n+1} v_{n-1} - u_{n+1}^2 v_n - u_n u_{n+2} v_n - u_n^2 u_{n+1} v_{n-1} v_n \\ &\quad - u_n u_{n+1}^2 v_n^2 - u_{n+1} u_{n+2} v_{n+1} - u_n u_{n+1} u_{n+2} v_n v_{n+1}, \\ G_2^{(1)} &= v_{n-2} + u_{n-1} v_{n-2} v_{n-1} + u_n v_{n-1}^2 + u_n v_{n-2} v_n + u_{n+1} v_{n-1} v_n \\ &\quad + u_{n-1} u_n v_{n-2} v_{n-1} v_n + u_n^2 v_{n-1}^2 v_n + u_n u_{n+1} v_{n-1} v_n^2, \\ G_1^{(2)} &= -u_{n-2} - u_{n-2} u_{n-1} v_{n-1} - u_{n-1}^2 v_n - u_{n-2} u_n v_n - u_{n-2} u_{n-1} u_n v_{n-1} v_n \\ &\quad - u_{n-1}^2 u_n v_n^2 - u_{n-1} u_n v_{n+1} - u_{n-1} u_n^2 v_n v_{n+1}, \\ G_2^{(2)} &= u_{n-1} v_n v_{n+1} + u_{n-1} u_n v_n^2 v_{n+1} + u_n v_{n+1}^2 + u_n^2 v_n v_{n+1}^2 \\ &\quad + v_{n+2} + u_n v_n v_{n+2} + u_{n+1} v_{n+1} v_{n+2} + u_n u_{n+1} v_n v_{n+1} v_{n+2}. \end{aligned}$$

4.6 – Generalized Toda lattices

In [36], the integrability of the chain

$$\ddot{y}_n = \dot{y}_{n+1} e^{(y_{n+1} - y_n)} - e^{2(y_{n+1} - y_n)} - \dot{y}_{n-1} e^{(y_n - y_{n-1})} + e^{2(y_n - y_{n-1})}, \tag{42}$$

which is related to the relativistic Toda lattice has been studied. With the change of variables, $u_n = \dot{y}_n,\ v_n = \exp(y_{n+1} - y_n)$, lattice (42) can be written as

$$\dot{u}_n = v_n(u_{n+1} - v_n) - v_{n-1}(u_{n-1} - v_{n-1}), \qquad \dot{v}_n = v_n(u_{n+1} - u_n). \tag{43}$$

Here, $u_n \sim v_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$. We computed a couple of symmetries for (43). One of them reads:

$$G_1 = u_{n-1}^2 v_{n-1} + u_{n-1} u_n v_{n-1} + u_{n-2} v_{n-2} v_{n-1} - v_{n-2}^2 v_{n-1} - 2u_{n-1} v_{n-1}^2$$

$$
\begin{aligned}
& -u_n v_{n-1}^2 + v_{n-1}^3 - u_n u_{n+1} v_n - u_{n+1}^2 v_n + u_n v_n^2 + 2u_{n+1} v_n^2 \\
& -v_n^3 - u_{n+2} v_n v_{n+1} + v_n v_{n+1}^2, \\
G_2 \;=\; & u_n^2 v_n - u_{n+1}^2 v_n + u_{n-1} v_{n-1} v_n - v_{n-1}^2 v_n - u_n v_n^2 + u_{n+1} v_n^2 \\
& -u_{n+2} v_n v_{n+1} + v_n v_{n+1}^2.
\end{aligned}
$$

In [37], Suris investigated the integrability of

$$
\ddot{y}_n = \dot{y}_n \left[\exp\left(y_{n+1} - y_n\right) - \exp\left(y_n - y_{n-1}\right) \right], \tag{44}
$$

which is closely related to the classical Toda lattice (22). The same change of variables as for (42) allows one to rewrite (44) as

$$
\dot{u}_n = u_n(v_n - v_{n-1}), \qquad \dot{v}_n = v_n(u_{n+1} - u_n). \tag{45}
$$

Again, $u_n \sim v_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$, and we have computed three symmetries. Two of them are:

$$
\begin{aligned}
G_1^{(1)} &= u_n^2 v_n + u_n u_{n+1} v_n + u_n v_n^2 - u_{n-1} u_n v_{n-1} - u_n^2 v_{n-1} - u_n v_{n-1}^2, \\
G_2^{(1)} &= u_{n+1}^2 v_n + u_{n+1} v_n^2 + u_{n+1} v_n v_{n+1} - u_n^2 v_n - u_n v_{n-1} v_n - u_n v_n^2, \\
G_1^{(2)} &= u_n^3 v_n + u_n^2 u_{n+1} v_n + u_n u_{n+1}^2 v_n + 2u_n^2 v_n^2 + 2u_n u_{n+1} v_n^2 + u_n v_n^3 \\
&\quad + u_n u_{n+1} v_n v_{n+1} - u_{n-1}^2 u_n v_{n-1} - u_{n-1} u_n^2 v_{n-1} - u_n^3 v_{n-1} \\
&\quad - u_{n-1} u_n v_{n-2} v_{n-1} - 2u_{n-1} u_n v_{n-1}^2 - 2u_n^2 v_{n-1}^2 - u_n v_{n-1}^3, \\
G_2^{(2)} &= u_{n+1}^3 v_n - u_n^3 v_n - u_{n-1} u_n v_{n-1} v_n - 2u_n^2 v_{n-1} v_n - u_n v_{n-1}^2 v_n - 2u_n^2 v_n^2 \\
&\quad + 2u_{n+1}^2 v_n^2 - u_n v_{n-1} v_n^2 - u_n v_n^3 + u_{n+1} v_n^3 + 2u_{n+1}^2 v_n v_{n+1} \\
&\quad + u_{n+1} u_{n+2} v_n v_{n+1} + u_{n+1} v_n^2 v_{n+1} + u_{n+1} v_n v_{n+1}^2.
\end{aligned}
$$

5 – The Integrability Package

We now briefly describe the use of our Integrability Package, which has (among other things) the code for the computation of symmetries based on the algorithms in Sections 2 and 3. The Integrability Package is written in *Mathematica* [39] syntax. Users are assumed to have access to *Mathematica* 3.0. All the necessary files are available in *MathSource* [15] including on-line help, documentation, and built-in examples. The corresponding files should be put in the appropriate places on your platform. Detailed instructions are given in the documentation.

After launching *Mathematica*, type

```
In[1]:= <<Integrability`
```

to read in the code. Doing so, you will get the following statement:

```
Loading init.m for Integrability from AddOns.
```

For the purpose of symmetry computations, the functions **PDESymmetries** and **DDESymmetries** are available.

Working with (32) as an example, the first two lines define the system ($r = q^*$), whereas the third line will produce the three symmetries listed in (33):

```
In[2]:= pde1:= D[q[x,t],t] - D[q[x,t],{x,2}] +
              2*q[x,t]^2*r[x,t] == 0;

In[3]:= pde2:= D[r[x,t],t] + D[r[x,t],{x,2}] -
              2*r[x,t]^2*q[x,t] == 0;

In[4]:= PDESymmetries[{pde1,pde2},{q,r},{x,t},{4,6},
            WeightRules->{Weight[q]->Weight[r]}]
```

Help about the functions and their options can be obtained by typing

```
In[5]:= ??DDESymmetries
DDESymmetries[eqn, u, {n, t}, R, opts] finds the symmetry with rank
   R of a differential-difference equation for the function u.
   DDESymmetries[{eqn1, eqn2, ...}, {u1, u2, ...}, {n, t}, R, opts]
   finds the symmetry of a system of differential-difference
   equations, where R is the rank of the first equation in the
   desired symmetry. DDESymmetries[{eqn1, eqn2, ...},
   {u1, u2, ...}, {n, t}, {Rmax}, opts] finds the symmetries with
   rank 0 through Rmax. DDESymmetries[{eqn1, eqn2, ...},
   {u1, u2, ...}, {n, t}, {Rmin, Rmax}, opts] finds the symmetries
   with rank Rmin through Rmax. n is understood as the discrete
   space variable and t as the time variable.

Attributes[DDESymmetries] = {Protected, ReadProtected}

Options[DDESymmetries] =
  {WeightedParameters -> {}, WeightRules -> {},
   MaxExplicitDependency -> 0, UndeterminedCoefficients -> C}
```

and

```
In[6]:= ??WeightedParameters
WeightedParameters is an option that determines the parameters with
   weight. If WeightedParameters -> {p1, p2, ...},
   then p1, p2, .... are considered as constant parameters with
   weight. The default is WeightedParameters -> {}.
```

```
Attributes[WeightedParameters] = {Protected}
```

The option **WeightedParameters** is useful when working with systems that lack uniformity in rank. In such cases, the code tries to resolve the problem of lack of uniformity, and prints appropriate messages. If the code can not automatically resolve the problem it suggests the use of the **WeightedParameters** option. Therefore, one should not use the option **WeightedParameters** unless it is explicitly suggested. For further descriptions of the functions and their options we refer to the documentation in [15].

6 – Other Software Packages

Higher-order symmetries can be computed with prolongation methods and numerous software packages are available that can aid in the tedious computations inherent to such methods. An extensive review of software for Lie symmetry computations, including generalized symmetries, can be found in [18, 19].

With prolongation methods one generates and subsequently reduces and solves a determining system of linear homogeneous partial differential equations for the unknown higher-order symmetry. In many cases, due to the length and complexity of that system, the general solution is out of reach and one resorts to making a polynomial *ansatz* for the symmetry.

Although restricted to polynomial higher-order symmetries, we believe that the method presented in this paper is much more straightforward. Furthermore, it does not require the application of prolongation methods or Lie algebraic techniques.

To avoid retreading the surveys [18, 19], here, we restrict our discussion to symbolic packages that allow one to compute generalized symmetries of PDEs, as they were defined in Section 2.1. We are not aware of software for DDEs to calculate the type of symmetries defined in Section 3.1.

Based on the alternative strategy discussed in Remark (ii) in Section 2, Ito's programs in REDUCE [22, 23] compute polynomial higher-order symmetries for systems of evolution equations that are uniform in rank (no weighted parameters can be introduced). Ito programs can not be used to computed symmetries that explicitly depend on x and t.

In [10], Fuchssteiner *et al.* present an algorithm to compute higher-order symmetries of evolution equations. Their algorithm is based on Lie algebraic techniques and uses commutator algebra on the Lie algebra of vector fields. Their approach is different from the usual prolongation method in that no determining equations are solved. Instead, all necessary generators of the finitely generated

Virasoro algebra are computed from one given element by direct Lie algebra methods. Their code is available in MuPAD.

The REDUCE program **FS** for "formal symmetries" was written by Gerdt and Zharkov [12]. The code **FS** can be applied to polynomial nonlinear PDEs of evolution type, which are linear with respect to the highest-order spatial derivatives and with non-degenerated, diagonal coefficient matrix for the highest derivatives. The algorithm in **FS** requires that the evolution equation are of order two or higher in the spatial variable. However, this approach does not require that the evolution equations are uniform in rank. With **FS** one cannot compute symmetries that depend explicitly on x and t.

The PC package **DELiA**, written in Turbo PASCAL by Bocharov [4] and co-workers, is a commercial computer algebra system for investigating differential equations using Lie's approach. The program deals with higher-order symmetries, conservation laws, integrability and equivalence problems. It has a special routine for systems of evolution equations. The program requires the presence of second or higher-order spatial derivative terms in all equations.

Finally, Sanders and Wang [33] have Maple and FORM software that aids in the computation of recursion operators.

7 – Conclusions

We have implemented direct algorithms (in *Mathematica*) that allow the user to compute polynomial higher-order symmetries of polynomial systems of evolution and lattice equations.

These algorithms are based on the dilation invariance of the given equations. Only minor modifications of our strategy lead to direct algorithms for conserved densities for systems of nonlinear PDEs [13] and nonlinear DDEs [14, 16].

For systems that arise from a variational principle, conservation laws follow from higher-order symmetries (Noether's theorem) and vice versa. Currently, in our algorithms we are not exploiting such connections.

In the future we will investigate generalizations of our methods to PDEs and DDEs in multiple space dimensions. We will also study the potential use of Lie-point symmetries other than dilation symmetries. Moreover, most recursion operators, which connect generalized symmetries, are also dilation invariant. We are extending our algorithms to the symbolic computation of recursion operators.

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References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[2] M. J. Ablowitz and J. F. Ladik, Nonlinear differential-difference equations and Fourier analysis, J. Math. Phys., 17 (1976) 1011–1018.

[3] M. J. Ablowitz and J. F. Ladik, A nonlinear difference scheme and inverse scattering, Stud. Appl. Math., 55 (1976) 213–229.

[4] A. V. Bocharov, *DELiA: a system for exact analysis of Differential Equations using S. Lie Approach*, DELiA 1.5.1 User Guide, Beaver Soft Programming Team, New York (1991).

[5] R. K. Dodd and J. D. Gibbon, The prolongation structure of a higher order Korteweg-de Vries equation, Proc. R. Soc. Lond. A, 358 (1977) 287–296.

[6] N. Euler and W. H. Steeb, *Continuous Symmetries, Lie Algebras and Differential Equations*, Wissenschaftsverlag, Mannheim, 1992.

[7] A. S. Fokas, A symmetry approach to exactly solvable evolution equations, J. Math. Phys., 21 (1980) 1318–1325.

[8] A. S. Fokas, Symmetries and integrability, Stud. Appl. Math., 77 (1987) 253–299.

[9] A. Fordy and J. Gibbons, Some remarkable nonlinear transformations, Phys. Lett. A, 75 (1980) 325.

[10] B. Fuchssteiner, S. Ivanov and W. Wiwianka, Algorithmic determination of infinite-dimensional symmetry groups for integrable systems in 1+1 dimensions, Mathl. Comput. Modelling, 25 (1997) 91–100.

[11] B. Fuchssteiner, W. Oevel and W. Wiwianka, Computer-algebra methods for investigation of hereditary operators of higher order soliton equations, Comput. Phys. Commun., 44 (1987) 47–55.

[12] V. P. Gerdt and A. Y. Zharkov, Computer generation of necessary integrability conditions for polynomial-nonlinear evolution systems, in *Proc. ISSAC '90*, S. Watenabe and M. Nagata eds., Academic Press, New York, 1990, pp. 250–254.

[13] Ü. Göktaş and W. Hereman, *Symbolic computation of conserved densities for systems of nonlinear evolution equations*, J. Symbolic Computation, 24 (1997) 591–621.

[14] Ü. Göktaş and W. Hereman, Computation of conservation laws for nonlinear lattices, Physica D (1998) to appear.

[15] Ü. Göktaş and W. Hereman, The software package and the related files are available at http://www.mathsource.com/cgi-bin/msitem?0208-932. *MathSource* is a vast electronic library of *Mathematica* material.

[16] Ü. Göktaş, W. Hereman and G. Erdmann, Computation of conserved densities for systems of nonlinear differential-difference equations, Phys. Lett. A, 236 (1997) 30-38.

[17] M. Hénon, Integrals of the Toda lattice, Phys. Rev. B, 9 (1974) 1921–1923.

[18] W. Hereman, Symbolic software for Lie symmetry analysis, in: CRC Handbook of Lie Group Analysis of Differential Equations, Volume 3: New Trends in Theoretical Developments and Computational Methods, Chapter 13, Ed.: N.H. Ibragimov, CRC Press, Boca Raton, Florida (1996) 367-413.

[19] W. Hereman, Review of symbolic software for Lie symmetry analysis, Mathl. Comput.

Modelling, 25 (1997) 115-132.
[20] R. Hirota and M. Ito, Resonance of solitons in one dimension, J. Phys. Soc. Jpn., 52 (1983) 744–748.
[21] R. Hirota and J. Satsuma, Soliton solutions of a coupled Korteweg-de Vries equation, Phys. Lett. A, 85 (1981) 407–408.
[22] M. Ito, A REDUCE program for finding symmetries of nonlinear evolution equations with uniform rank, Comput. Phys. Commun., 42 (1986) 351–357.
[23] M. Ito, SYMCD - a REDUCE package for finding symmetries and conserved densities of systems of nonlinear evolution equations, Comput. Phys. Commun., 79 (1994) 547–554.
[24] M. Kac and P. van Moerbeke, On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices, Adv. Math., 16 (1975) 160–169.
[25] D. J. Kaup, On the inverse scattering problem for cubic eigenvalue problems of the class $\Psi_{xxx} + 6Q\Psi_x + 6R\Psi = \lambda\Psi$, Stud. Appl. Math., 62 (1980) 189–216.
[26] B. A. Kupershmidt and G. Wilson, Modifying Lax equations and the second Hamiltonian structure, Invent. Math., 62 (1981) 403–436.
[27] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Commun. Pure Appl. Math., 21 (1968) 467–490.
[28] A. V. Mikhailov, A. B. Shabat and V. V. Sokolov, The symmetry approach to classification of integrable equations, in *What Is Integrability?*, V. E. Zakharov ed., Springer-Verlag, Berlin Heidelberg, 1991, pp. 115–184.
[29] P. J. Olver, Evolution equations possessing infinitely many symmetries, J. Math. Phys., 18 (1977) 1212–1215.
[30] P. J. Olver, *Applications of Lie Groups to Differential Equations,* 2nd Edition, Springer Verlag, New York, 1993.
[31] A. Ramani, B. Grammaticos and K. M. Tamizhmani, An integrability test for differential-difference systems, J. Phys. A: Math. Gen., 25 (1992) L883–L886.
[32] V. Rosenhaus, G. H. Katzin, On symmetries, conservation laws, and variational problems for partial differential equations, J. Math. Phys., 35 (1994) 1998–2012.
[33] J. Sanders and J. P. Wang, On hereditary recursion operators, Report WS-472, Department of Mathematics and Computer Sciences, Free University, Amsterdam, The Netherlands, Physica D (1997) submitted.
[34] J. Satsuma and D. J. Kaup, A Bäcklund transformation for a higher order Korteweg-De Vries equation, J. Phys. Soc. Jpn., 43 (1977) 692–697.
[35] K. Sawada and T. Kotera, A method for finding N-Soliton solutions of the K.d.V. equation and K.d.V.-like equation, Prog. Theor. Phys., 51 (1974) 1355–1367.
[36] Y. B. Suris, New integrable systems related to the relativistic Toda lattice, J. Phys. A: Math. Gen., 30 (1997) 1745–1761.
[37] Y. B. Suris, On some integrable systems related to the Toda lattice, J. Phys. A: Math. Gen., 30 (1997) 2235–2249.
[38] M. Toda, *Theory of nonlinear lattices*, Springer Verlag, Berlin, 1981.
[39] S. Wolfram, *The Mathematica book*, 3rd Edition, Wolfram Media, Urbana-Champaign, Illinois & Cambridge University Press, London, 1996.