QUEUEING NETWORKS VIA MATHEMATICA

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Queueing networks are used as a main tool for modeling and analysis of many engineering applications. These applications include for example computer systems, communication networks and manufacturing systems. Classical models of queueing networks are not suitable for molding modern communication networks and manufacturing systems. Some new models were formulated to overcome the limitations existing in classical models. These models are re-entrant lines and multiclass queueing networks. Analytical analysis of such networks is hard and they do not have (till now) a closed form solution for the steady state distribution for the functionals under investigation. Thus, another technique was proposed to obtain the performance measures. This is the performance bounds technique. Such technique appeared in the literature for Markovian networks where the bounds were obtained by constructing a linear program and solving it. The construction was done manually and there was a need for automation. We designed a Mathematica package to automate the process by relying on the symbolic power of Mathematica to generate the constraints. Then, using a built in function in Mathematica, the linear programming problem is solved. Moreover, we extended the technique to non-Markovian cases by providing an adequate mathematical theory. Moreover, the automation in this case rely on the symbolic power of Mathematica due to the large size of the problem. The technique was also taken to a different setting namely discrete time queueing networks which models some parts that lie in the heart of today's Internet. Again a Mathematica package was developed for such purpose. Finally, to compare the obtained results we used some existing methods for comparison as well as simulation results by using a Mathematica package for simulation. In this article, we give an overview of the technical results as well as sample of the outputs of some examples solved using the packages. We are in the process of turning these packages into a comprehensive one to deal with queueing analysis for beginners and researchers. A Mathematica notebook is provided as an appendix which contains outputs of some of the packages.

1. INTRODUCTION

Queueing networks are commonly used as a tool for the modeling and analysis of many engineering applications. More specifically, queueing networks are used to model situations that require providing service in more than one place of the system where a competition exists for the required service. These situations include for example: computer systems [9], communication networks [6] and manufacturing systems [5].

The classical models of queueing networks such as Jackson network [8] and Kelly network [10] have an important property that enables the analytical analysis of such networks. More specifically, these networks have a product form solution for the steady state distribution of the number of customers in the various nodes of the network. By a product form solution, it is meant that the required distribution factorizes into a product form where each term depends on the specifications of a single node of the network. From this distribution, performance measures of the network such as expected total number of customers and expected delay time can be calculated.

Classical models of queueing networks have limitations that prevent them of being suitable for modeling modern communication networks and manufacturing systems. In such systems, it is required to build policies that depend on the customer class. Moreover, it is desired to give different service rates to different customer classes that compete for the service at a specific node. These are two examples of features that are not permitted in classical queueing networks models and which are important in modern engineering applications.

Some new queueing networks models were formulated to overcome the limitations existing in the classical models. For example, the re-entrant lines [11] model was introduced to make a suitable abstraction of semiconductor manufacturing systems. The multiclass queueing networks model [4] is a more general model that includes the re-entrant lines as a special case.

Although the new developed models are more suitable for the modeling and analysis purpose of modern engineering applications, their analytical analysis is harder than the classical models. They have not (till now) a closed form solution for the steady state distribution of the number of customers in the various nodes of the network and it is doubtful that such distribution can be obtained. Thus, some other techniques were proposed to obtain the performance measures of such models. The technique in which we are interested here is the performance bounds technique.

2. PERFORMANCE BOUNDS TECHNIQUE AND AUTOMATION PROCESS

Since the exact value of the performance measures is not available in the new formulated models, it may be sufficient from a practical point of view to obtain upper and lower bounds on such performance measures. If the bounds are tight enough, an approximate value of the performance measure is obtained. Moreover, if one is interested in computing the total delay through the network, the lower bound serves as the optimal value of that performance measure. In other words, the lower bound is the performance measure value when the system is working under the optimal policy. Other policies can be compared with this value to test their nearness to the optimal policy.

The performance bounds are obtained by constructing a linear program whose solution (for both minimization and maximization) gives the required bounds. The objective function is the required performance measure (mainly the total number of customers) and the constraints are obtained by assuming stability and examining the consequence of the steady state on general quadratic forms.

When the performance bounds technique was introduced [12], the linear program was generated manually and it was mentioned that there is a need for automation the solution process. We performed this automation. A Mathematica package was written such that only the specifications of the network are given and then the symbolic manipulation power of Mathematica is used to generate the required linear program. The resulting linear program is then solved for both minimization and maximization to obtain the required bounds.

3. PERFORMANCE BOUNDS FOR NON-MARKOVIAN NETWORKS

We extended the mathematical theory of the performance bounds technique to include more general systems. More specifically, the performance bounds technique was introduced to deal with Markovian queueing networks [12]. We extended the technique to deal with networks having Erlangian service times and interarrival times. The developed Mathematica package was also extended to generate the required bounds for these general cases.

The developed package was used to examine several numerical examples. Part of the results is shown in Figure 1 (See for more details [1], [15]). The bounds on the total number of customers are computed for the re-entrant line of Figure 2. In the first plot of Figure 1, it was assumed that arrivals follow a Poisson process and the service times follow an Erlangian distribution while in the second plot, the interarrival times follow an Erlangian distribution and the service times are exponentially distributed. Since this class of networks does not possess a closed form solution, the obtained bounds were compared with simulation results. In fact, we used Mathematica to perform this simulation. From the examined examples, it appeared that the technique provides tight bounds in the light and medium loading. However, in the heavy loading case, there was a rapid increase in the upper bound. We concluded that additional set of constraints is required to bring the bounds tight enough in the heavy loading conditions.

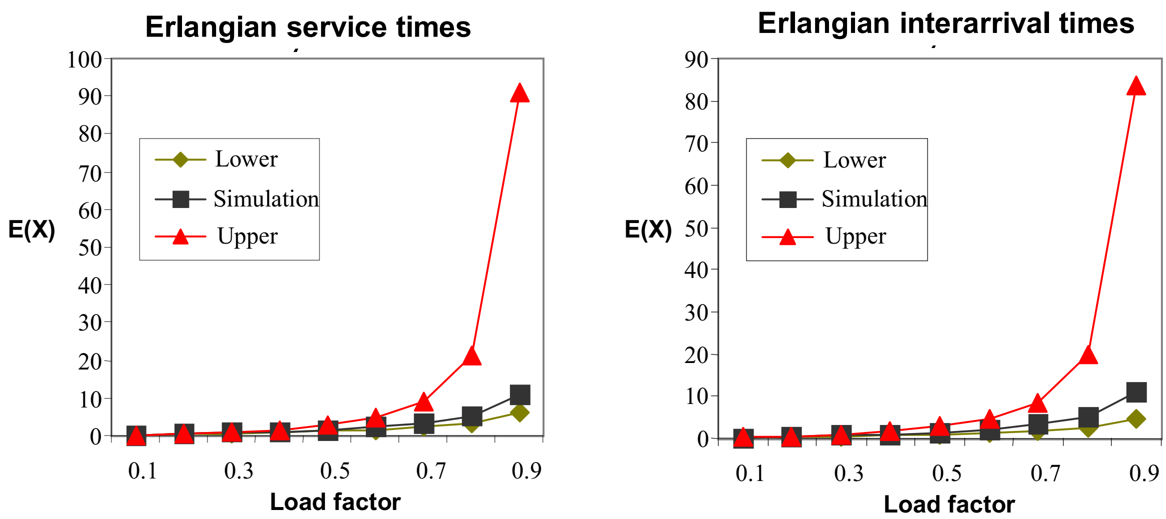

Figure 1 Upper and lower bounds on the total number of customers in the re-entrant line of Figure 2.

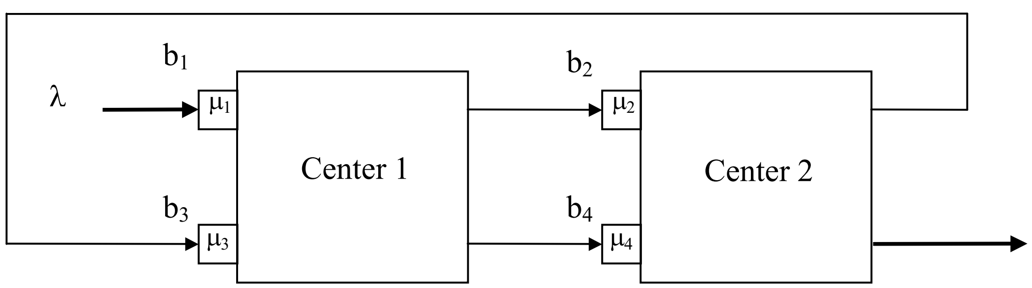

Figure 2 A re-entrant line example.

4. PERFORMANCE BOUNDS FOR DISCRETE TIME QUEUEING NETWORKS

We adapted [16] the performance bounds technique to deal with discrete time queueing systems [7]. In a discrete time queueing system, the time axis is divided into intervals of equal length and the system events (arrival and departures) are allowed to occur only at the boundaries of these intervals. Discrete time queueing systems are currently used to model communication networks working under the ATM technology [13], [14], [17] which lies in the heart of today's Internet.

We considered discrete time Jackson network [3]. In the continuous time case, Jackson network possesses the independence property [8], i.e., the number of customers at any node is independent of the number of customers at other nodes. This important property was proved to be unavailable in the discrete time case [3]. Hence, it is hard to obtain a closed form solution for this network in the discrete time case. The performance bounds technique provides a suitable alternative solution.

The performance bounds technique was adapted to deal with discrete time Jackson network. A Mathematica package was developed to automate the solution process. This package was used to evaluate a large number of examples [15]. We considered two types of arrival streams: Bernoulli and Poisson. Moreover, we studied the case where the arrival rates at all nodes are the same as well as the other case where every node has a different arrival rate. Finally, three routing mechanisms were assumed:

1. Uniform routing where the served customer leaves the network or joins any other node with the same probability.
2. Symmetric routing where the served customer at node i joins node j with the same probability with which served customer at node j joins node i.
3. Neighborhood routing where the served customer joins the next node or leaves the network with the same probability and he can't join any other node.

The interested reader is referred to [15] for more insights.

Part of the results is shown in Figure 3. In these plots, we see that the bounds are very tight in the case of equal arrival rates and uniform or symmetric routing. Moreover, in these cases the difference between the bounds approaches rapidly a certain limit as the load factor increases to its upper limit. This phenomena doesn't exist in the other cases. This phenomena is very important since it implies that the bounds are tight also in the heavy traffic case. The worst case (described by a rapid increase in the upper bound) appears in the neighborhood routing mechanism which is the nearest case to the tandem network.

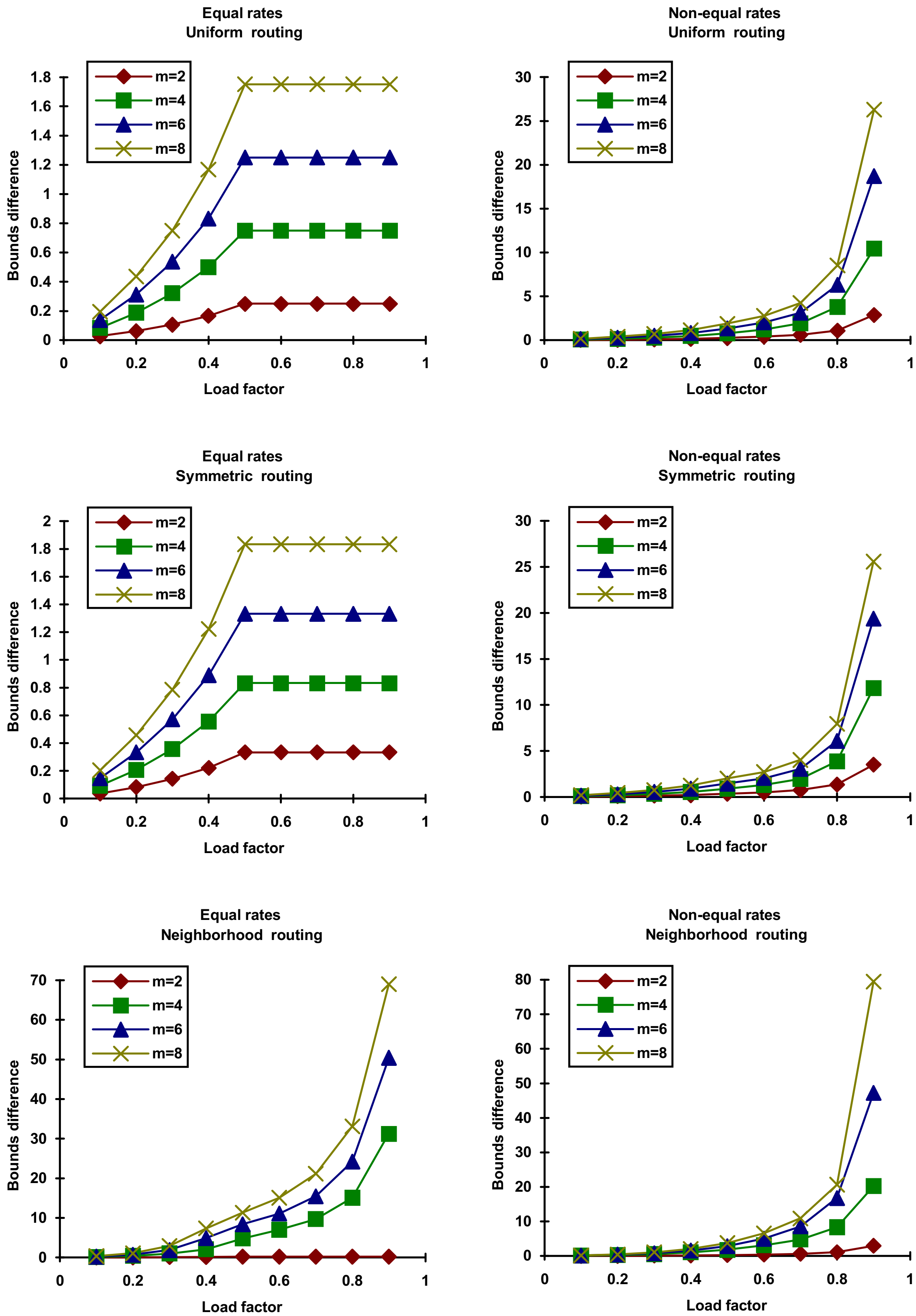

Figure 3 Bounds tightness in the discrete time Jackson network with Bernoulli arrivals.

From these examples (and from many others we have tested), it can be said that our bounds are very tight when the arrival rates are equal and each node is connected to every other node. Moreover, in this class of networks the bounds are tight enough for all load factors including the heavy traffic case. When the arrival rates are not equal or when some nodes can't be reached from other nodes, the bounds become less tight especially in the heavy traffic loads. The worst case appears in the tandem queueing system.

4. CONCLUSION

In this article, we described very briefly an extensive work in performance analysis of queueing networks where, what is called performance bounds technique, was extended and developed. We used Mathematica to automate the process of getting the sought results relying on the symbolic and numerical power of Mathematica. Moreover, an extension of the performance bounds technique which has been developed mathematically to the non-Markovian case was given. From the examined examples, it appeared that the technique provides tight bounds in the light and medium loading while heavy loading requires an additional set of constraints to bring the bounds more tight. The performance bounds technique was taken to a different setting namely discrete time queueing networks which are fundamental in today's communication networks. The developed Mathematica package that deals with the discrete time case was used to examine several examples. From these examples, a class of networks with very bounds was identified. It was shown that this tightness property is still valid in the heavy traffic case. In fact, we are in the process of turning the developed packages into a comprehensive one that deals with queueing analysis for beginners and researchers as well as enhancing the packages to deal with queueing networks with more added features.

APPENDIX

A Mathematica notebook of sample results

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