Generalized symmetries

A Mathematica *program to determine Lie–Bäcklund symmetries*

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Abstract. In the following we shall demonstrate the application of the genaralized symmetry method on partial as well as on ordinary differential equations. The method of deriving generalized symmetries is reviewed and the construction of integrals of motion is demonstrated. The conditions of integrability follow as side conditions from the determining equations for the generalized characteristics in which cases the corresponding integrals of motion are derivable. The generalized symmetry analysis is supported by a *Mathematica* program called `LieBaecklund[]`.

1 Introduction

At the end of the last century, Lie considered the invariance of differential equations under the transformation of dependent and independent variables [1]. Lie was able to classify and solve some types of ordinary as well as partial differential equations. Although very general, his original work mainly considered the geometric aspects of the transformations related to the equations. Essentially, he considered invariance under point transformations and demonstrated that some sort of contact transformations are related to point transformations in case of first order partial differential equations. However, it turns out that much wider classes of transformations leave differential equations invariant including those considered by Lie. Emmi Noether [2] and Felix Klein [3] demonstrated that a system of differential equations derivable from a variational principle allows a much greater set of transformations under which a given system of differential equations is invariant. They significantly extended the application of symmetry group methods by including derivatives of the dependent variables in the transformations. Moreover, they were able to offer a regular procedure to construct the related conservation laws which are based on the investigation of the invariance properties of the variational integral under the action of transformation groups. Since Noether published her work, it is well known that Lie-type point symmetries cannot provide for the complete symmetry group underlying a given system of differential equations. Consequently, the required integrals of motion and the solutions are not available for many systems.

In recent years, the symmetry methods have become more attractive, especially in the field of nonlinear dynamics [4, 5]. In this article, we will consider

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a generalized method to determine the symmetries for nonlinear partial (PDE) and ordinary (ODE) differential equations. We will demonstrate that the method is applicable on models such as the diffusion equation, the Burger equation, the Hénon-Heiles model, two dimensional quartic oscillators relevant in the description of light propagation in a fiber, and the motion of two ions in a Paul or Penning trap. In section 2 and 3, we will outline the mathematical procedure used in *Mathematica* programs based on the Lie group approach. In section 8, we will give some results for the above models, generalized symmetries and integrals of motion. Finally, some concluding remarks are given in section 9.

2 Generalized symmetries

The setting of our program in *Mathematica* is an entirely general one. Its algorithm is well known and described e.g. in the books by Olver [5] or Bluman and Kumei [4]. We consider the general case of a nonlinear system of differential equations for an arbitrary number q of unknown functions u^a which may depend on p independent variables x_i. We denote these sets of variables simply by $u = (u^1, u^2, ..., u^q)$ and $x = (x_1, x_2, ..., x_p)$, respectively. The general case is given by a system of m nonlinear differential equations

$$\Delta^i(x, u^{(k)}) = 0 \qquad i = 1, 2, ..., m \tag{1}$$

of order k. The term $u^{(k)}$ is understood as k-th derivative of u with respect to x. We note that m, k, p and q are arbitrary, positive integers.

In order to find out some of the properties of (1), it is suitable to apply a transformation of the independent, and dependent variables as well as the derivatives of the dependent variables to determine the attributes of the corresponding group G. This procedure is now available for many computers and can be done by the program `LieBaecklund[]` written in *Mathematica*.

First let us consider transformations of (1) depending on a single parameter ϵ which are given by

$$x' = \Xi\ (x, u^{(k)}; \epsilon) \tag{2}$$

$$u' = \Phi\ (x, u^{(k)}; \epsilon) \tag{3}$$

with $k = 0, 1, 2\ldots$ denoting the derivational order. The functions Ξ and Φ are assumed to be differentiable with respect to the dependent and independent varaibles. The identity transformation of the coordinates in (2,3) is given by $\epsilon = 0$. In analogy to Lie's theory, the transformations (2,3) are generated by the infinitesimal operator

$$\vec{v} = \sum_{i=1}^{p} \xi_i(x, u^{(k)}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}. \tag{4}$$

which today we call a vector field related to equations (1) [5]. The infinitesimals ξ_i and ϕ_α are derivable from (2) and (3) by differentiating the transformations with respect to the group parameter ϵ. Considering these expressions for $\epsilon = 0$, we get

$$\xi_i(x, u^{(k)}) = \left. \frac{\partial \Xi(x, u^{(k)}; \epsilon)}{\partial \epsilon} \right|_{\epsilon=0} \tag{5}$$

$$\phi_\alpha(x, u^{(k)}) = \left. \frac{\partial \Phi_\alpha(x, u^{(k)}; \epsilon)}{\partial \epsilon} \right|_{\epsilon=0} . \tag{6}$$

The infinitesimal transformation corresponding to (2 and 3) is then given by

$$x_i' = x_i + \epsilon \xi_i(x, u^{(k)}) + O(\epsilon^2) \qquad i = 1, 2, \ldots, p \tag{7}$$

$$u'^\alpha = u^\alpha + \epsilon \phi_\alpha(x, u^{(k)}) + O(\epsilon^2) \qquad \alpha = 1, 2, \ldots, q. \tag{8}$$

with ϵ being the group parameter and a small quantity. This representation is a straight generalization of Lie's procedure. On the other hand it is sufficient to use only the infinitesimal transformation in the form

$$x_i' = x_i \qquad i = 1, 2, \ldots, p \tag{9}$$

$$u'^\alpha = u^\alpha + \epsilon \left\{ \phi_\alpha(x, u^{(k)}) - \sum_{i=1}^{p} u_i^\alpha \xi_i(x, u^{(k)}) \right\} + O(\epsilon^2) \tag{10}$$

$$\alpha = 1, 2, \ldots, q.$$

This means that the complete transformation properties are representable in a transformation of the dependent variables [4]. The corresponding representation of the vector field $\vec{v}_Q$ is then

$$\vec{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha} \tag{11}$$

where $Q_\alpha = Q_\alpha(x, u^{(k)})$ is the characteristic of the vectorfield $\vec{v}_Q$. The characteristics Q_α are related to the infinitesimals by

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi_i u_i^\alpha \qquad \alpha = 1, 2, \ldots, q. \tag{12}$$

It is obvious from these relations that point symmetries are a subset of the generalized symmetries. The determination of the characteristics follows by a similar algorithm as used in the determination of point symmetries [6]. The determining equations for the characteristics Q_α are consequences of the relation

$$pr\,\vec{v}_Q\,(\Delta)|_{D_J\Delta=0} = 0 \tag{13}$$

where $pr\,\vec{v}_Q$ denotes the prolongation of the vectorfield $\vec{v}_Q$. The general expression of the prolongation reads

$$pr\,\vec{v}_Q = \sum_{\alpha=1}^{p}\sum_J D_J(Q_\alpha)\frac{\partial}{\partial u_J^\alpha} \tag{14}$$

with D_J the total derivative related to the multindex J.

3 Generalized symmetries and consrvation laws

A great number of problems in physics can be described by second order ordinary differential equations

$$\Delta_k(t, \{q_j\}, \{\dot{q}_j\}, \{\ddot{q}_j\}) = 0 \qquad k, j = 1, 2, \ldots, N \tag{15}$$

where $\{q_j\} = \{q_j(t)\}$ are the set of the dependent variables. Dots denote differentiation with respect to the independent variable t. Δ_k are given functions of dependent and independent variables which for example can be considered as Newton's or Lagrange's equations.

Let us assume that (15) can be derived from a generating function of the S action by applying the variational calculus and by making a variation of the individual paths q_j's in

$$S = \int_{t_1}^{t_2} \mathcal{L}(t, \{q_j\}, \{\dot{q}_j\})dt. \tag{16}$$

We well know that under these assumptions the system of equations (15) are the Euler-Lagrange equations $E_k(\mathcal{L}) = \Delta_k = \frac{\delta S}{\delta q_k} = 0$, $k = 1, 2, \ldots, N$ derivable by the variation of the path q_k. The symbol $\frac{\delta}{\delta q_k} = \sum_{n=0}^{\infty}(-1)^n \frac{d^n}{dt^n}\frac{\partial}{\partial q_k^{(n)}}$ denotes the variational derivative or the Euler operator. At this preliminary stage of our examination, we know the equations of motion and probably the physical interpretation contained in the equations. It is, however, unknown to us under which conditions solutions for the system (15) can be found.

3.1 Generalized symmetries

As mentioned above, Lie's original procedure $pr^{(2)}\vec{v}\Delta_k)|_{\Delta_k=0} = 0$, $k = 1, 2, \ldots, N$ to determine symmetries of ODE's does not provide all symmetries of the equations. In turn, we have to use a generalized method to uncover the 'hidden' symmetries. A generalization of the point transformations can be obtained, if the infinitesimals τ and ϕ_j do not only depend on the dependent and independent variables but also on derivatives of the q_j's. In case of second order differential equations, only the space of coordinates by the first derivatives needs to be extended. All higher derivatives can be eliminated by the differential equations and its derivatives. To denote the extended dependencies including derivatives, we use square brackets $[q] = (t, \{q_j\}, \{\dot{q}_j\})$ in the infinitesimals $\tau = \tau[q]$, and $\phi_j = \phi_j[q]$.

The corresponding generalized vector field for such transformations can be formally given by

$$\vec{v} = \tau[q]\frac{\partial}{\partial t} + \sum_{j=1}^{N} \phi_j[q]\frac{\partial}{\partial q_j}. \tag{17}$$

In a second step, we can introduce so-called evolutionary symmetries which are related to the generalized symmetries by $Q_j[q] = \phi_j[q] - \tau[q]\dot{q}_j$. The corresponding vector field has the representation

$$\vec{v}_Q = \sum_{j=1}^{N} Q_j[q]\frac{\partial}{\partial q_j}. \tag{18}$$

The symmetries contained in (17) and (18) are essentially the same [5]. The only difference is that (18) considers transformations of the dependent variables and derivatives in the extended form while (17) additionally takes into account a transformation of the independent variables. The main adventage of the evolutionary representation is its simplicity in the representation of the prolongation

$$\mathbf{pr}\vec{v}_Q = \sum_{j=1}^{N}\sum_{J=0}^{\infty} \frac{d^J}{dt^J}\left(Q_j[q]\right)\frac{\partial}{\partial q_j^{(J)}}. \tag{19}$$

The assumption in the calculation of the related symmetries of (15) is the same as in the case of point symmetries. Thus, the computation of the generalized symmetries proceeds in essentially the same way as the computation of Lie point symmetries. Since we consider the variables and derivatives as independent up to a fixed order, we not only take equations (15) for themselves into account but also consider the derivatives of (15) as concerns the invariance condition. The invariance condition remains nearly the same and reads

$$\mathbf{pr}\vec{v}_Q\left(\Delta_k\right)\Big|_{\frac{d^J}{dt^J}\Delta_k=0} = 0 \qquad \text{with} \qquad J = 0, 1, 2, \ldots \quad \text{and} \quad k = 1, 2, \ldots, N. \tag{20}$$

Contrary to point symmetries, differential equations can be used to eliminate higher derivatives and also the differential consequences of equations (15). Equation (20) provides the determining equations for the characteristics Q_j. Though linear, it is very complicated to solve this system of equations as it yields a large overdetermined system of equations. With some skill, solutions of (20) can be found by some ad hoc ansatz for the Q_j's; for example, by assuming that Q_j has a special dependence (powers, polynomials, etc.) on one or some of its arguments. By this method, equation (20) will split into several equations that may or may not have a nonzero solution. In our *Mathematica* program `LieBaecklund[]`, we use for second order ODEs a polynomial ansatz in the coordinates and momenta. If we can solve the huge system of linear equations in the coefficients introduced by the ansatz, we will have determined the characteristics up to some power in the variables q_j and momenta $\dot{q}_j$. This information can in turn, be used to construct conservation laws and invariant solutions. For partial differential equations, `LieBaecklund[]` delivers a simplified system of linear partial differential equations determining the characteristics Q_j's.

3.2 Conservation laws

With equation (16), we assumed that the system of equations (15) is derivable by a variation of the path q_j. In close analogy to the question of invariance of (15), we can ask for the invariance of the action itself. This means that we have to assume

$$\int_{t_1}^{t_2} \mathcal{L}(t, \{q_j\}, \{\dot{q}_j\})dt = \int_{t_1^*}^{t_2^*} \mathcal{L}(t^*, \{q_j^*\}, \{\dot{q}_j^*\})dt^*. \tag{21}$$

Two questions arise at this points **i)** Are the group elements $g \in G$ of (21) derivable from infinitesimal conditions? **ii)** How are the symmetries of the Euler-Lagrange equations related to these variational symmetries? It is obvious that local transformations preserving the invariance of a functional transform one solution of the variational problem into another. This must be the case if relation (21) is correct. Thus, we can use the Lie procedure applied to the Euler-Lagrange equations corresponding to (16), taking into account that not all symmetries of the equations (15) are symmetries of (16). To uncover the variational symmetries, we can use the methods discussed so far but need to check in addition the relation [5]

$$\mathbf{pr}\vec{v}_Q\left(\Delta_k\right) + D_Q^*\left(\Delta_k\right) = 0 \qquad k = 1, 2, \ldots, N \tag{22}$$

where D_Q^* denotes the adjoint Fréchet derivative of the characteristic Q. The evolutionary vector field $\vec{v}_Q$ with its characteristics Q is a variational symmetry if equation (22) holds for all t and q_j's. We now have a criterion singling out these symmetries from the generalized symmetries which are also variational symmetries. Noethers theorem establishes the connection between the variational symmetries with their characteristics Q_j and the conserved quantities. The case of one independent variable can be summarized in the formula

$$\frac{d}{dt} I = \sum_{j=1}^{N} Q_j \Delta_j = 0 \tag{23}$$

which is a special case of the more general divergence formula given by Olver [5]. The essential points in determining integrals of motion for a given Euler-Lagrange system are i) to find the characteristics Q_j for the variational symmetries, and ii) to integrate equation (23). These two, are steps among other implemented in the *Mathematica* function `LieBaecklund[]`. To show how the outlined analysis can be used in practical applications, we will give some examples demonstrating these *Mathematica* functions.

4 Algorithm used in Mathematica

The algorithm presented here follows the notation used by Olver [5]. The work of Olver is usable in a direct implementation for a computer program. Using his notations the algorithm involves the following steps:

i) Applying the prolongation operator $pr\vec{v}_Q$ to each equation $\Delta^i(x, u^{(k)})$ and requiring that equation (13) is satisfied. The prolongation of the vector field $\vec{v}_Q$ in (11) is determined by means of the formula (14).

ii) Consider (1) as an algebraic system of equations where derivatives become new variables. Provided this system of algebraic equations can be solved by the *Mathematica* function `Solve[]`, we can gain m solutions w^β which contain the rest of the equations Δ^i. The variables w^β with $\beta = 1, 2, \ldots, m$ and derivational consequences of these relations are used as substitution rules or side conditions in equation (13).

iii) Obtaining the determining equations for the characteristics Q_α by equating to zero the coefficients of all functionally independent expressions in the remaining derivatives u^α_J.

iv) Simplifying the total number of equations by using the first order derivatives and homogeneous higher order derivatives for the characteristics. This step is repeated until the number of determining equations reach a certain stable value.

v) If the determining equations in case of ordinary differential equations are known it may be possible to solve these equations by using a polynomial ansatz for the characteristics. We note that these step will not provide the most general solution if such one exists but delivers at least a subclass of it.

This procedure works expediently as long as the variables w^β are obtainable by `Solve[]` from the equations of motion (1) and if `Solve[]` can find a solution from the linear equations for the constants used in the ansatz of the characteristics. The manual procedure, feeding back some information on the characteristics gained by partly solving the determining equations is completely independent from the automatic solution procedure. If one is convinced that the ansatz used in `LieBaecklund[]` does not provide the general most solution one can use this manual and iterative procedure to find the solutions of the determining equations.

4.1 Description of the program `LieBaecklund[]`

We used *Mathematica* 2.0 for the `LieBaecklund[]` program. This is a version of *Mathematica* which is available for many computers ranging from a Pc up to a mainframe computer. Information concerning the definition and principal features of *Mathematica* can be found in S. Wolfram's handbook [15].

The Program essentially consists of function definitions and uses options to controll the execution of the calculation. In fact, an appropriate function definition is given in the program for each task. The `LieBaecklund[]` function is the main program. Once called, it reads the input–data from an input file and goes through the algorithm by sequentially calling the necessary functions.

4.2 Options of `LieBaecklund[]`

The calculational steps of `LieBaecklund[]` are mainly controled by the following options:

- `AnsatzPoly->{0,0}`: `AnsatzPoly` contains two numbers in a list. The first number gives the polynomial degree of the Ansatz with respect to the coordinates. The second integer determines the polynomial degree with respect to the first derivatives. Default values of `AnsatzPoly` is a list containing two zeros {0,0}. If `AnsatzPoly` contains integers greater than zero a polynomial ansatz for the characteristics is created and used in the calculation of the integrals.

- `CharExpression->{}`: Contains a list of explicit or implicit expressions for the characteristics Q_α. As default no information is given on the characteristics.

- `DerivativeOrder->1`: The order of derivatives taken into account for the dependencies of the charactersitics. The default value of `DerivativeOrder` is 1.

- `VariationalCheck->False`: If set to `True` a check of the variational condition is performed. If this check is satisfied the corresponding integrals of motion are calculated

- `Warning->True`: warnings are given if nonnumeric terms are eliminated in the determining equations; default value is `True`.

5 Input Data

The information needed by the program to perform the computation is contained in an input file. The contence of this file is written in conventional *Mathematica* notation. This file is read by `LieBaecklund[]`; provided, it exists in the current directory. If the file does not exist `LieBaecklund[]` is terminated with an error message stating that the filename given is not available in the current directory. The input file contains:

1. A list of independent variables with the name: `IndepVar`. The sequential order of the independent variables in this list should represent the sequential order of the variables used as arguments of the dependent variables.

2. A list of dependent variables with the name: `DependVar`

3. The left hand side of the equations of motion (1). The equations $\Delta^i, i = 1, ..., m$ must be given in real form in the list `EqList`. All dependencies and differentiations must be explicitly written out.

4. The list `parameteR` contains parameters of the model which are treated as constants.

5. The variables w^β for the side conditions $|_{\Delta=0}$ in (13) are collected in the list `SubsList`. This list should contain the terms for which `EqList` can be solved by the *Mathematica* function `Solve`.

6. List `ListChar` contains information about the characteristics.

A sample set of such input data is included in the online description of function `LieBaecklund[]`. Calling for help with `?LieBaecklund` at the *Mathematica* prompt gives you a short descrition of the options and the input file. Contrary to other programs in REDUCE, MACSYMA, etc., the equations Δ^i are not converted to an artificial representation, denoting differentiation and dependent variables by a vector of cumbersome numbers. The input data of `LieBaecklund[]` are in a form actually used in pencil calculations. The representation of equations can be used in the form used in handwritten notation, i.e. $\frac{\partial u(x,t)}{\partial ,t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0$ corresponds in *Mathematica* to `D[u[x,t]t]- D[u[x,t],{x,2}]`. The expressions used in the substitution list `SubsList` and the characteristics list `ListChar` must be of the same explicite form. The input data is put into a file with the use of *Mathematica* assignment statements. This file is read by the program if `LieBaecklund[]` is called with the file name as first argument.

6 Output Data

If `LieBaecklund[]` finished the computation in a regular way some information are printed on the screen. During the calculation results and messages concerning the state of the calculation are displayed. Some samples are given in the section Exaples. The results produced by `LieBaecklund[]` are also available for further manipulations in the list `FinalResults`.

The determining equations are printed on the screen and are available for further manipulation in the list `FinalResultes`. While the program is running, the total output is simultaneously logged in a file whos first part of the name is derived from the name used in calling `LieBaecklund[]` but its four rightmost characters are replaced by `.lba`.

7 Examples for partial differential equations

7.1 The one dimensional diffusion equation

As a first example, let us consider the generalized symmetries of the diffusion equations [5, 4]. In order to obtain the determining system, we need to apply the prolongation $pr\vec{v}_Q$ according to (13) to the diffusion equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0. \tag{24}$$

Equation (24) and some information on the dependencies and substitutions are given in the listing of the input file named `diffu.lbd`. To read the data stored in this file, the program is called with `LieBaecklund["diffu.lbd"]`. This command initiates the calculation of the determining equations and writes the logging to a file called `diff.lba`.

```
(* ---    Diffusion Equation      --- *)
(*----------------------------------*)
(* --- independent variables --- *)
IndepVar={x,t};
(* --- dependent variables --- *)
DependVar={u};
(* --- equations  --- *)
EqList = {D[u[x,t],t] - D[u[x,t],{x,2}]}
(* --- parameters --- *)
parameteR = {}
(* --- substitutions --- *)
SubsList = {D[u[x,t],t]};
(* --- lists of the characteristics --- *)
ListChar = {D[u[x,t],x]};
```

```
Date :  {1993, 3, 17, 15, 6, 16}
Loading data from  diffu.lbd
```

```
-------------------------------------------------------------------
Equation No. 1

 (0,1)          (2,0)
u     [x, t] - u     [x, t] = 0

-------------------------------------------------------------------

Substitution No. 1

 (0,1)           (2,0)
u     [x, t] -> u     [x, t]

Substitution No. 2

 (1,1)           (3,0)
u     [x, t] -> u     [x, t]

-------------------------------------------------------------------
 Determinig the Substitutions                     0.93 Second
 Creating the Characteristics                     0.16 Second
 Calculate generalized prolongations              0.44 Second
 Separate numerator and denominator               0.22 Second
 Substitute the higher order derivatives          0.11 Second
 Determine the coefficients of the prolongation 1.05 Second
 Entering simplification
 The number of equations before simplification
 *** N_before: 3


 The number of equations after simplification
 *** N_after: 3

 Simplification finished                          0.6 Second


Equation No. : 1
-------------------------------------------------------------------
   (0,0,0,2)                (1,0)
QC5          [x, t, u[x, t], u     [x, t]] = 0

-------------------------------------------------------------------

Equation No. : 2
-------------------------------------------------------------------
   (1,0)      2    (0,0,2,0)                 (1,0)
-(u     [x, t]  QC5          [x, t, u[x, t], u     [x, t]]) +

      (0,1,0,0)                (1,0)
>    QC5          [x, t, u[x, t], u     [x, t]] -

      (1,0)          (1,0,1,0)                 (1,0)
>    2 u     [x, t] QC5          [x, t, u[x, t], u     [x, t]] -
```

```
            (2,0,0,0)                  (1,0)
>     QC5          [x, t, u[x, t], u      [x, t]] = 0

   --------------------------------------------------------------------

   Equation No. : 3
   --------------------------------------------------------------------
       (1,0)          (0,0,1,1)                  (1,0)
   -2 u      [x, t] QC5         [x, t, u[x, t], u      [x, t]] -

           (1,0,0,1)                  (1,0)
>     2 QC5          [x, t, u[x, t], u      [x, t]] = 0

   --------------------------------------------------------------------

   **** The determining equations are stored in FinalResults.  ****
```

The result is a set of three determining equations for the characteristic Q_α named `QC5` in the output. To verify that a solution of these equations is given by the expression $Q_1 = u_x$ we call `LieBaecklund[]` by

```
LieBaecklund["diffu.lbd",CharExpression->ListChar]
```

The verificatio of the solution took about 2 sec on a 486 Pc.

```
  Date :  {1993, 3, 17, 15, 8, 10}
  Loading data from  diffu.lbd


   --------------------------------------------------------------------
   Equation No. 1

    (0,1)          (2,0)
   u      [x, t] - u      [x, t] = 0

   --------------------------------------------------------------------

   Substitution No. 1

    (0,1)           (2,0)
   u      [x, t] -> u      [x, t]

   Substitution No. 2

    (1,1)           (3,0)
   u      [x, t] -> u      [x, t]

   --------------------------------------------------------------------
   --------------------------------------------------------------------
   Characteristics :
```

```
      (1,0)
Q1 = u     [x, t]
--------------------------------------------------------------------
 Determinig the Substitutions                      0.82 Second
 Creating the Characteristics                      0.05 Second
 Calculate generalized prolongations               0.44 Second
 Separate numerator and denominator                0.16 Second
 Substitute the higher order derivatives           0.05 Second
 Determine the coefficients of the prolongation 0.94 Second
 Entering simplification
 The number of equations before simplification
 *** N_before: 3


 The number of equations after simplification
 *** N_after: 3

 Simplification finished                           0.55 Second
 Inserting the given information                   0.38 Second


**** The determining equations are stored in FinalResults.  ****
--------------------------------------------------------------------
                      Equations satisfied!
--------------------------------------------------------------------
```

7.2 The Burger equation

Another example for determining generalized symmetries is the Burger equation. The input file is given below

```
(* ---      Burger Equation       --- *)
(*-----------------------------------*)
(* --- independent variables --- *)
IndepVar={x,t};
(* --- dependent variables --- *)
DependVar={u};
(* --- equations  --- *)
EqList = {D[u[x,t],t] - D[u[x,t],{x,2}] - D[u[x,t],x]^2}
(* --- parameters --- *)
parameteR = {}
(* --- substitutions --- *)
SubsList = {D[u[x,t],t]};
(* --- characteristics --- *)
ListChar = c1;

c1 = {Derivative[1, 0][u][x, t]}
```

The derivation of the determining equations reads

```
Date :  {1993, 3, 17, 15, 9, 23}
Loading data from  burger.lbd

 -------------------------------------------------------------------
 Equation No. 1

  (0,1)          (1,0)      2    (2,0)
 u     [x, t] - u     [x, t]  - u     [x, t] = 0

 -------------------------------------------------------------------

 Substitution No. 1

  (0,1)           (1,0)      2    (2,0)
 u     [x, t] -> u     [x, t]  + u     [x, t]

 Substitution No. 2

  (1,1)             (1,0)        (2,0)          (3,0)
 u     [x, t] -> 2 u     [x, t] u     [x, t] + u     [x, t]

 -------------------------------------------------------------------
  Determinig the Substitutions                   0.66 Second
  Creating the Characteristics                   0.05 Second
  Calculate generalized prolongations            0.5 Second
  Separate numerator and denominator             0.21 Second
  Substitute the higher order derivatives        0.16 Second
  Determine the coefficients of the prolongation 1.1 Second
  Entering simplification
  The number of equations before simplification
  *** N_before: 3


  The number of equations after simplification
  *** N_after: 3

  Simplification finished                        0.61 Second


 Equation No. : 1
 -------------------------------------------------------------------
    (0,0,0,2)                (1,0)
 QC7          [x, t, u[x, t], u     [x, t]] = 0

 -------------------------------------------------------------------

 Equation No. : 2
 -------------------------------------------------------------------
    (1,0)      2    (0,0,1,0)               (1,0)
 -(u     [x, t]  QC7          [x, t, u[x, t], u     [x, t]]) -

     (1,0)      2    (0,0,2,0)               (1,0)
>    u     [x, t]  QC7          [x, t, u[x, t], u     [x, t]] +
```

```
        (0,1,0,0)                (1,0)
>    QC7         [x, t, u[x, t], u     [x, t]] -

        (1,0)          (1,0,0,0)                (1,0)
>    2 u     [x, t] QC7         [x, t, u[x, t], u     [x, t]] -

        (1,0)          (1,0,1,0)                (1,0)
>    2 u     [x, t] QC7         [x, t, u[x, t], u     [x, t]] -

        (2,0,0,0)                (1,0)
>    QC7         [x, t, u[x, t], u     [x, t]] = 0

  --------------------------------------------------------------------

  Equation No. : 3
  --------------------------------------------------------------------
      (1,0)          (0,0,1,1)                (1,0)
  -2 u     [x, t] QC7         [x, t, u[x, t], u     [x, t]] -

          (1,0,0,1)                (1,0)
>    2 QC7         [x, t, u[x, t], u     [x, t]] = 0

  --------------------------------------------------------------------

  **** The determining equations are stored in FinalResults.  ****
```

The verification of the characteristics by the command

```
LieBaecklund["burger.lbd",CharExpression->ListChar]
```

gives

```
 Date :  {1993, 3, 17, 15, 10, 38}
 Loading data from  burger.lbd


  --------------------------------------------------------------------
  Equation No. 1

   (0,1)          (1,0)      2    (2,0)
  u     [x, t] - u     [x, t]  - u     [x, t] = 0

  --------------------------------------------------------------------

  Substitution No. 1

   (0,1)           (1,0)      2    (2,0)
  u     [x, t] -> u     [x, t]  + u     [x, t]

  Substitution No. 2
```

```
 (1,1)              (1,0)        (2,0)          (3,0)
u     [x, t] -> 2 u     [x, t] u     [x, t] + u     [x, t]

-------------------------------------------------------------------
-------------------------------------------------------------------
Characteristics :

     (1,0)
Q1 = u     [x, t]
-------------------------------------------------------------------
 Determinig the Substitutions                    0.88 Second
 Creating the Characteristics                    0.05 Second
 Calculate generalized prolongations             0.5 Second
 Separate numerator and denominator              0.16 Second
 Substitute the higher order derivatives         0.17 Second
 Determine the coefficients of the prolongation 1.16 Second
 Entering simplification
 The number of equations before simplification
 *** N_before: 3


 The number of equations after simplification
 *** N_after: 3

 Simplification finished                         0.6 Second
 Inserting the given information                 0.5 Second


**** The determining equations are stored in FinalResults.  ****
-------------------------------------------------------------------
                      Equations satisfied!
-------------------------------------------------------------------
```

8 Models of second order ODEs

In this section, we will deal with the application of the generalized symmetry method for ordinary differential equations of second order previously introduced. We discuss the Hénon–Heiles model, a quartic anharmonic oscillator, and present results for the two ion problem in a Paul trap. We will describe how the method works and where its shortcommings come into play. All calculations are supported by the two *Mathematica* programs `Lie[]` [6] and `LieBaecklund[]`.

8.1 The Hénon-Heiles model

The Hénon-Heiles model for gravitating stars in a cylindrical galaxy [13] is given by

$$\Delta = \begin{pmatrix} \ddot{q}_1 + Aq_1 + 2Dq_1q_2 \\ \ddot{q}_2 + Bq_2 + Dq_1^2 - Cq_2^2 \end{pmatrix} = 0 \qquad (25)$$

where A, B, C and D are constants which determine the dynamical behavior of equation (25). Given an arbitrary choice of parameters, chaotic evolution occurs for q_1 and q_2 above a critical energy level [13, 14]. The non-chaotic cases are given in table I where also the generalized symmetries and integrals of motion are listed. These results are obtainable by the command

```
LieBaecklund["henon.lbd",AnsatzPoly->{1,1}]
```

which reads the input data from the file `henon.lbd` containing the starting information for the calculation. A listing of this file is given below.

```
(**********************************************************************
*                                                                     *
* --- Henon Heiles ---                                                *
*                                                                     *
**********************************************************************)
(* --- independent variables --- *)
IndepVar={t};

(* --- dependent variables --- *)
DependVar={q1,q2};

(* --- equations  --- *)
EqList = {
        D[q1[t],{t,2}] + a*q1[t] + 2*d*q1[t]*q2[t],
        D[q2[t],{t,2}] + b*q2[t] + d*q1[t]^2 -c*q2[t]^2};

(* --- parameter list --- *)
parameteR = {a,b,c,d};

(* --- substitutions --- *)
SubsList = {D[q1[t],{t,2}],D[q2[t],{t,2}]};

(* --- Characteristics --- *)
ListChar = {D[q1[t],t],D[q2[t],t]};
```

The results of the calculation read

```
Date :  {1993, 3, 17, 14, 26, 19}
Loading data from  henon.lbd


 -------------------------------------------------------------------
 Equation No. 1

 a q1[t] + 2 d q1[t] q2[t] + q1''[t] = 0

 Equation No. 2
```

```
        2                    2
d q1[t]  + b q2[t] - c q2[t]  + q2''[t] = 0

--------------------------------------------------------------------

Substitution No. 1

q1''[t] -> -(a q1[t]) - 2 d q1[t] q2[t]

Substitution No. 2

                    2                      2
q2''[t] -> -(d q1[t] ) - b q2[t] + c q2[t]

Substitution No. 3

  (3)
q1   [t] -> -(a q1'[t]) - 2 d q2[t] q1'[t] - 2 d q1[t] q2'[t]

Substitution No. 4

  (3)
q2   [t] -> -2 d q1[t] q1'[t] - b q2'[t] + 2 c q2[t] q2'[t]

--------------------------------------------------------------------
 Determinig the Substitutions                    1.49 Second
 Creating the Characteristics                    0.05 Second
 Calculate generalized prolongations             1.49 Second
 Separate numerator and denominator              1.15 Second
 Substitute the higher order derivatives         1.53 Second
 Determine the coefficients of the prolongation 15.16 Second
 Entering simplification
 The number of equations before simplification
 *** N_before: 2


 The number of equations after simplification
 *** N_after: 2

 Simplification finished                         2.08 Second
 Make an Ansatz for ODE's of second order
   create an Ansatz :                            0.38 Second
   substitute the Ansatz :                       7.36 Second
   resulting equations from the Ansatz :         3.35 Second
   calling Solve with 19 equations and        15 variables
   determine Coefficients of the Ansatz :        116.99 Second

 Check variational symmetry
   Variationalcheck finished                     47.24 Second


Characteristic No. : 1
--------------------------------------------------------------------
{0, q2'[t]}
```

```
Parameter: {a, b, c, d} = {a, a, 0, 0}

--------------------------------------------------------------------
           2          2
    a q2[t]    q2'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 2
--------------------------------------------------------------------
{q1'[t], 0}

Parameter: {a, b, c, d} = {a, a, 0, 0}

--------------------------------------------------------------------
           2          2
    a q1[t]    q1'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 3
--------------------------------------------------------------------
{0, q2'[t]}

Parameter: {a, b, c, d} = {a, b, c, 0}

--------------------------------------------------------------------
           2           3          2
    b q2[t]    c q2[t]    q2'[t]
I = -------- - -------- + -------
       2          3          2

====================================================================

Characteristic No. : 4
--------------------------------------------------------------------
{q1'[t], 0}

Parameter: {a, b, c, d} = {a, b, c, 0}

--------------------------------------------------------------------
           2          2
    a q1[t]    q1'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 5
--------------------------------------------------------------------
{q1'[t], q2'[t]}
```

```
Parameter: {a, b, c, d} = {a, b, c, d}

--------------------------------------------------------------------
           2          2                2          3         2         2
    a q1[t]           2     b q2[t]    c q2[t]    q1'[t]    q2'[t]
I = -------- + d q1[t]  q2[t] + -------- - -------- + ------- + -------
       2                           2          3          2         2

====================================================================

Characteristic No. : 6
--------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a, b, c, d} = {a, b, -6 d, d}

--------------------------------------------------------------------
           2          2                2                      2         2
    a q1[t]           2     b q2[t]            3   q1'[t]    q2'[t]
I = -------- + d q1[t]  q2[t] + -------- + 2 d q2[t]  + ------- + -------
       2                           2                       2         2

====================================================================

Characteristic No. : 7
--------------------------------------------------------------------
  -4 a   b
  ---- + -
   d     d                   q1[t] q2'[t]  -(q1[t] q1'[t])
{(-------- + q2[t]) q1'[t] - ------------, ---------------}
     4                            2               2

Parameter: {a, b, c, d} = {a, b, -6 d, d}

--------------------------------------------------------------------
             4         2      2                          2      2
    -(d q1[t] )   q1[t]  (-4 a  + a b - 4 a d q2[t] - 4 d  q2[t] )
I = ----------- + ------------------------------------------------ +
         8                              8 d

                                  2
>    (-4 a + b + 4 d q2[t]) q1'[t]    q1[t] q1'[t] q2'[t]
     ------------------------------ - -------------------
                  8 d                          2

====================================================================

Characteristic No. : 8
--------------------------------------------------------------------
{0, q2'[t]}

Parameter: {a, b, c, d} = {a, a, 0, 0}

--------------------------------------------------------------------
```

```
          2         2
    a q2[t]    q2'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 9
--------------------------------------------------------------------
{q1'[t], 0}

Parameter: {a, b, c, d} = {a, a, 0, 0}

--------------------------------------------------------------------
          2         2
    a q1[t]    q1'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 10
--------------------------------------------------------------------
{0, q2'[t]}

Parameter: {a, b, c, d} = {a, a, 0, 0}

--------------------------------------------------------------------
          2         2
    a q2[t]    q2'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 11
--------------------------------------------------------------------
{q1'[t], 0}

Parameter: {a, b, c, d} = {a, a, 0, 0}

--------------------------------------------------------------------
          2         2
    a q1[t]    q1'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 12
--------------------------------------------------------------------
{0, q2'[t]}

Parameter: {a, b, c, d} = {a, b, 0, 0}
```

```
--------------------------------------------------------------------
            2          2
    b q2[t]     q2'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 13
--------------------------------------------------------------------
{q1'[t], 0}

Parameter: {a, b, c, d} = {a, b, 0, 0}

--------------------------------------------------------------------
            2          2
    a q1[t]     q1'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 14
--------------------------------------------------------------------
{0, q2'[t]}

Parameter: {a, b, c, d} = {a, a, c, 0}

--------------------------------------------------------------------
            2           3          2
    a q2[t]     c q2[t]     q2'[t]
I = -------- - -------- + -------
       2           3          2

====================================================================

Characteristic No. : 15
--------------------------------------------------------------------
{q1'[t], 0}

Parameter: {a, b, c, d} = {a, a, c, 0}

--------------------------------------------------------------------
            2          2
    a q1[t]     q1'[t]
I = -------- + -------
       2          2

====================================================================

Characteristic No. : 16
--------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a, b, c, d} = {a, a, c, d}
```

```
--------------------------------------------------------------------
           2                           2          3         2         2
    a q1[t]           2         a q2[t]    c q2[t]    q1'[t]    q2'[t]
I = -------- + d q1[t]  q2[t] + -------- - -------- + ------- + -------
       2                           2          3          2         2

====================================================================

Characteristic No. : 17
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                 -2 d
Parameter: {a, b, c, d} = {a, b, ----, d}
                                  5

--------------------------------------------------------------------
           2                           2            3         2         2
    a q1[t]           2         b q2[t]    2 d q2[t]    q1'[t]    q2'[t]
I = -------- + d q1[t]  q2[t] + -------- + ---------- + ------- + -------
       2                           2           15          2         2

====================================================================

Characteristic No. : 18
--------------------------------------------------------------------
{0, q2'[t]}

Parameter: {a, b, c, d} = {a, 4 a, 0, 0}

--------------------------------------------------------------------
                       2
             2   q2'[t]
I = 2 a q2[t]  + -------
                    2

====================================================================

Characteristic No. : 19
--------------------------------------------------------------------
                q1[t] q2'[t]  -(q1[t] q1'[t])
{q2[t] q1'[t] - ------------, ---------------}
                     2               2

Parameter: {a, b, c, d} = {a, 4 a, 0, 0}

--------------------------------------------------------------------
             2                          2
    -(a q1[t]  q2[t])   q2[t] q1'[t]    q1[t] q1'[t] q2'[t]
I = ----------------- + ------------ - -------------------
            2                 2                  2

====================================================================
```

```
Characteristic No. : 20
-------------------------------------------------------------------
{q1'[t], 0}

Parameter: {a, b, c, d} = {a, 4 a, 0, 0}

-------------------------------------------------------------------
           2         2
    a q1[t]    q1'[t]
I = -------- + -------
       2          2

===================================================================

Characteristic No. : 21
-------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a, b, c, d} = {a, a, -6 d, d}

-------------------------------------------------------------------
           2                            2                      2         2
    a q1[t]             2        a q2[t]             3   q1'[t]    q2'[t]
I = -------- + d q1[t]  q2[t] + -------- + 2 d q2[t]  + ------- + -------
       2                            2                      2         2

===================================================================

Characteristic No. : 22
-------------------------------------------------------------------
  -3 a                    q1[t] q2'[t]  -(q1[t] q1'[t])
{(---- + q2[t]) q1'[t] - ------------, ---------------}
  4 d                          2               2

Parameter: {a, b, c, d} = {a, a, -6 d, d}

-------------------------------------------------------------------
             4         2     2                     2      2
    -(d q1[t] )   q1[t]  (3 a  + 4 a d q2[t] + 4 d  q2[t] )
I = ----------- - ----------------------------------------- +
         8                          8 d

                               2
   (-3 a + 4 d q2[t]) q1'[t]     q1[t] q1'[t] q2'[t]
>   -------------------------- - -------------------
              8 d                         2

===================================================================

Characteristic No. : 23
-------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a, b, c, d} = {a, a, -d, d}
```

```
--------------------------------------------------------------------
              2                          2          3         2         2
    a q1[t]            2         a q2[t]    d q2[t]    q1'[t]    q2'[t]
I = -------- + d q1[t]  q2[t] + -------- + -------- + ------- + -------
       2                            2          3         2         2

====================================================================

Characteristic No. : 24
--------------------------------------------------------------------
{q2'[t], q1'[t]}

Parameter: {a, b, c, d} = {a, a, -d, d}

--------------------------------------------------------------------
            3
    d q1[t]
I = -------- + q1[t] q2[t] (a + d q2[t]) + q1'[t] q2'[t]
       3

====================================================================

Characteristic No. : 25
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                  -2 d
Parameter: {a, b, c, d} = {a, a, ----, d}
                                   5

--------------------------------------------------------------------
              2                          2            3         2         2
    a q1[t]            2         a q2[t]    2 d q2[t]    q1'[t]    q2'[t]
I = -------- + d q1[t]  q2[t] + -------- + ---------- + ------- + -------
       2                            2          15          2         2

====================================================================

**** The characteristics are stored in FinalResults. ****
```

The results which do not alter the original equations of motion (25) are collected in table I.

8.2 Two dimensional quartic oscillators

Some physical models from nonlinear optics [7] and accelerator physics, for example, can be formulated in a standard form for a coupled system of anharmonic oscillators

$$\Delta = \begin{pmatrix} \ddot{q}_1 + \frac{\alpha_1}{2} q_1 - 2\Gamma_1 q_1^3 - 2 q_1 q_2^2 \\ \ddot{q}_2 + \frac{\alpha_2}{2} q_2 - 2\Gamma_2 q_2^3 - 2 q_2 q_1^2 \end{pmatrix} = 0 \tag{26}$$

TABLE I
The Hénon-Heiles model

Parameters	Q_1	Q_2	Integral
A, B, C, D	$\dot{q}_1$	$\dot{q}_2$	$\frac{1}{2}(p_1^2+p_2^2)+\frac{A}{2}q_1^2+\frac{B}{2}q_2^2+Dq_1^2q_2-\frac{C}{3}q_2^3$
$A = B,\ D = -C$	$\dot{q}_2$	$\dot{q}_1$	$\dot{q}_1\dot{q}_2 + \frac{D}{3}q_1^3 + Dq_1q_2^2 + Aq_1q_2$
$A, B,\ C = -6D$	$-2q_2\dot{q}_1 + \frac{1}{2D}(4A - B)\dot{q}_1 + q_1\dot{q}_2$	$q_1\dot{q}_1$	$\dot{q}_1(q_1\dot{q}_2 - q_2\dot{q}_1) + \frac{4A-B}{4D}\dot{q}_1^2 + \frac{D}{4}q_1^4 + +Dq_1^2q_2^2 + Aq_1^2q_2 + \frac{A(4A-B)}{4D}q_1^2$
$B = 16A,\ C = -16D$	$(2Dq_2q_1^2 + Aq_1^2)\dot{q}_1 - \frac{1}{3}Dq_1^3\dot{q}_2 + \dot{q}_1^3$	$-\frac{1}{3}Dq_1^3\dot{q}_1$	$\frac{1}{4}\dot{q}_1^4+\left(\frac{A}{2}q_1^2 + Dq_1^2q_2\right)\dot{q}_1^2-\frac{D}{3}q_1^3\dot{q}_1\dot{q}_2- -\frac{D^2}{18}q_1^6 - \frac{D^2}{3}q_1^4q_2^2 - \frac{AD}{3}q_1^4q_2 + \frac{A^2}{4}q_1^4$

where α_i and Γ_i determine the strength of quadratic and quartic terms in the harmonic and anharmonic part of the potential, respectively. The integrals of motion, generalized symmetries, and parameter combinations are given in table II. The results given in table II results follow by the command

`LieBaecklund["4oscill.lbd",AnsatzPoly->{2,1}`.

A listing of such a calculation is given below.

```
Date :  {1993, 3, 17, 14, 37, 43}
Loading data from  4oscill.lbd


 ------------------------------------------------------------------
 Equation No. 1

 a1 q1[t]             3                2
 -------- - 2 G1 q1[t]  - 2 q1[t] q2[t]  + q1''[t] = 0
    2

 Equation No. 2

 a2 q2[t]          2                   3
 -------- - 2 q1[t]  q2[t] - 2 G2 q2[t]  + q2''[t] = 0
    2

 ------------------------------------------------------------------
```

```
Substitution No. 1

                                   3                 2
           -(a1 q1[t] - 4 G1 q1[t]  - 4 q1[t] q2[t] )
q1''[t] -> ------------------------------------------
                                2

Substitution No. 2

                                2                   3
           -(a2 q2[t] - 4 q1[t]  q2[t] - 4 G2 q2[t] )
q2''[t] -> ------------------------------------------
                                2

Substitution No. 3

  (3)                                2                 2
q1   [t] -> -(a1 q1'[t] - 12 G1 q1[t]  q1'[t] - 4 q2[t]  q1'[t] -

>        8 q1[t] q2[t] q2'[t]) / 2

Substitution No. 4

  (3)                                                      2
q2   [t] -> -(-8 q1[t] q2[t] q1'[t] + a2 q2'[t] - 4 q1[t]  q2'[t] -

                     2
>        12 G2 q2[t]  q2'[t]) / 2

--------------------------------------------------------------------
 Determinig the Substitutions                     2.08 Second
 Creating the Characteristics                     0.05 Second
 Calculate generalized prolongations              1.42 Second
 Separate numerator and denominator               2.31 Second
 Substitute the higher order derivatives          1.76 Second
 Determine the coefficients of the prolongation 19.67 Second
 Entering simplification
 The number of equations before simplification
 *** N_before: 2


 The number of equations after simplification
 *** N_after: 2

 Simplification finished                          2.64 Second
 Make an Ansatz for ODE's of second order
   create an Ansatz :                             0.55 Second
   substitute the Ansatz :                        14.22 Second
   resulting equations from the Ansatz :          14.11 Second
   calling Solve with 46 equations and         22 variables
   determine Coefficients of the Ansatz :         827.12 Second

 Check variational symmetry
   Variationalcheck finished                      104.63 Second
```

```
Characteristic No. : 1
-------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a1, a2, G1, G2}

-------------------------------------------------------------------
            2           4                2       2          4         2
    a1 q1[t]    G1 q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = --------- - --------- + --------------------- - --------- + ------- +
        4           2                 4                 2          2

         2
   q2'[t]
>  -------
      2

===================================================================

Characteristic No. : 2
-------------------------------------------------------------------
{q1'[t], q2'[t]}

                                      a2
Parameter: {a1, a2, G1, G2} = {--, a2, G1, G2}
                                      4

-------------------------------------------------------------------
            2           4                2       2          4         2
    a2 q1[t]    G1 q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = --------- - --------- + --------------------- - --------- + ------- +
       16           2                 4                 2          2

         2
   q2'[t]
>  -------
      2

===================================================================

Characteristic No. : 3
-------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a2, a2, G1, G2}

-------------------------------------------------------------------
            2           4                2       2          4         2
    a2 q1[t]    G1 q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = --------- - --------- + --------------------- - --------- + ------- +
        4           2                 4                 2          2

         2
   q2'[t]
```

```
>    -------
        2

  ====================================================================

  Characteristic No. : 4
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

  Parameter: {a1, a2, G1, G2} = {4 a2, a2, G1, G2}

  --------------------------------------------------------------------
                             4                 2       2           4          2
               2   G1 q1[t]     (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
  I = a2 q1[t]   - --------- + --------------------- - --------- + ------- +
                       2                 4                  2          2

           2
     q2'[t]
>    -------
        2

  ====================================================================

  Characteristic No. : 5
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

  Parameter: {a1, a2, G1, G2} = {a1, a2, 1, G2}

  --------------------------------------------------------------------
               2        4                  2       2           4          2
      a1 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
  I = --------- - ------ + --------------------- - --------- + ------- +
          4          2               4                  2          2

           2
     q2'[t]
>    -------
        2

  ====================================================================

  Characteristic No. : 6
  --------------------------------------------------------------------
        2
  {q2[t]  q1'[t] - q1[t] q2[t] q2'[t],

                                    -a1 + a2        2
>    -(q1[t] q2[t] q1'[t]) + (-------- + q1[t] ) q2'[t]}
                                       2

  Parameter: {a1, a2, G1, G2} = {a1, a2, 1, 1}

  --------------------------------------------------------------------
```

```
                         2                  4        2       2
      a2 (-a1 + a2) q2[t]    (a1 - a2) q2[t]    q2[t]  q1'[t]
  I = ------------------- + ---------------- + -------------- -
               8                   4                  2

                                                   2
>   q1[t] q2[t] q1'[t] q2'[t] + (-a1 + a2) q2'[t]
                                ------------------ +
                                        4

          2           2           2           2
>   q1[t]  (a1 q2[t]   - a2 q2[t]   + 2 q2'[t] )
    -----------------------------------------
                       4

  ===================================================================

  Characteristic No. : 7
  -------------------------------------------------------------------
  {q1'[t], q2'[t]}

  Parameter: {a1, a2, G1, G2} = {a1, a2, 1, 1}

  -------------------------------------------------------------------
               2        4                 2       2        4         2         2
      a1 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]    q2[t]    q1'[t]    q2'[t]
  I = --------- - ------ + ---------------------- - ------ + ------- + -------
          4         2                4                 2        2         2

  ===================================================================

  Characteristic No. : 8
  -------------------------------------------------------------------
  {q1'[t], q2'[t]}

  Parameter: {a1, a2, G1, G2} = {0, 0, G1, G2}

  -------------------------------------------------------------------
                4                                4         2         2
      -(G1 q1[t] )        2      2   G2 q2[t]    q1'[t]    q2'[t]
  I = ------------ - q1[t]  q2[t]  - --------- + ------- + -------
           2                             2          2         2

  ===================================================================

  Characteristic No. : 9
  -------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                  a2       1
  Parameter: {a1, a2, G1, G2} = {--, a2, -, G2}
                                  4        6

  -------------------------------------------------------------------
               2        4                 2       2           4         2
```

```
      a2 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
  I = --------- - ------ + ---------------------- - --------- + ------- +
         16         12                4                   2          2

           2
     q2'[t]
>    -------
        2

  ====================================================================

  Characteristic No. : 10
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                           1
  Parameter: {a1, a2, G1, G2} = {a2, a2, -, G2}
                                           3

  --------------------------------------------------------------------
              2        4                2       2           4         2
      a2 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
  I = --------- - ------ + ---------------------- - --------- + ------- +
          4         6                 4                   2          2

           2
     q2'[t]
>    -------
        2

  ====================================================================

  Characteristic No. : 11
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                 a2
  Parameter: {a1, a2, G1, G2} = {--, a2, 1, G2}
                                 4

  --------------------------------------------------------------------
              2        4                2       2           4         2
      a2 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
  I = --------- - ------ + ---------------------- - --------- + ------- +
         16         2                 4                   2          2

           2
     q2'[t]
>    -------
        2

  ====================================================================

  Characteristic No. : 12
  --------------------------------------------------------------------
```

```
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a2, a2, 1, G2}

--------------------------------------------------------------------
             2        4                 2       2           4         2
    a2 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = --------- - ------ + ---------------------- - --------- + ------- +
        4         2                4                  2          2

           2
     q2'[t]
>    -------
        2

====================================================================

Characteristic No. : 13
--------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {4 a2, a2, 1, G2}

--------------------------------------------------------------------
                      4                 2       2           4         2
            2   q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = a2 q1[t]  - ------ + ---------------------- - --------- + ------- +
                  2                4                  2          2

           2
     q2'[t]
>    -------
        2

====================================================================

Characteristic No. : 14
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                                  8
Parameter: {a1, a2, G1, G2} = {4 a2, a2, -, G2}
                                                  3

--------------------------------------------------------------------
                        4                 2       2           4         2
            2   4 q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = a2 q1[t]  - -------- + ---------------------- - --------- + ------- +
                   3                 4                  2          2

           2
     q2'[t]
>    -------
        2
```

```
=====================================================================

Characteristic No. : 15
---------------------------------------------------------------------
                    q1[t] q2'[t]  -(q1[t] q1'[t])
{q2[t] q1'[t] - ------------, ---------------}
                         2                2

                                        a2      1  8
Parameter: {a1, a2, G1, G2} = {--, a2, -, -}
                                        4       6  3

---------------------------------------------------------------------
            4             2                        2              2
     q1[t]  q2[t]    q1[t]  q2[t] (-3 a2 + 16 q2[t] )   q2[t] q1'[t]
I = ----------- + ------------------------------- + ------------ -
         6                         48                       2

     q1[t] q1'[t] q2'[t]
>    -------------------
              2

=====================================================================

Characteristic No. : 16
---------------------------------------------------------------------
{q1'[t], q2'[t]}

                                        a2      1  8
Parameter: {a1, a2, G1, G2} = {--, a2, -, -}
                                        4       6  3

---------------------------------------------------------------------
                2        4                 2       2        4        2
     a2 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]    4 q2[t]    q1'[t]
I = --------- - ------ + --------------------- - -------- + ------- +
        16         12               4                  3         2

           2
     q2'[t]
>    -------
        2

=====================================================================

Characteristic No. : 17
---------------------------------------------------------------------
{q1'[t], q2'[t]}

                                                1  1
Parameter: {a1, a2, G1, G2} = {a2, a2, -, -}
                                                3  3

---------------------------------------------------------------------
                2        4                 2       2        4        2        2
```

```
    a2 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]    q2[t]    q1'[t]    q2'[t]
I = --------- - ------ + ---------------------- - ------ + ------- + -------
        4         6                4                 6        2         2

====================================================================

Characteristic No. : 18
--------------------------------------------------------------------
{q2'[t], q1'[t]}

                                         1  1
Parameter: {a1, a2, G1, G2} = {a2, a2, -, -}
                                         3  3

--------------------------------------------------------------------
            3                                  2
    -2 q1[t]  q2[t]   q1[t] q2[t] (3 a2 - 4 q2[t] )
I = --------------- + ---------------------------- + q1'[t] q2'[t]
           3                        6

====================================================================

Characteristic No. : 19
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                           8  1
Parameter: {a1, a2, G1, G2} = {4 a2, a2, -, -}
                                           3  6

--------------------------------------------------------------------
                          4                2       2         4         2
            2    4 q1[t]    (a2 - 4 q1[t] ) q2[t]    q2[t]    q1'[t]
I = a2 q1[t]   - -------- + ---------------------- - ------ + ------- +
                    3                 4                12        2

        2
   q2'[t]
>  -------
      2

====================================================================

Characteristic No. : 20
--------------------------------------------------------------------
{q2[t] q2'[t], q2[t] q1'[t] - 2 q1[t] q2'[t]}

                                           8  1
Parameter: {a1, a2, G1, G2} = {4 a2, a2, -, -}
                                           3  6

--------------------------------------------------------------------
            3      2
    -2 q1[t]  q2[t]
I = ---------------- + q2[t] q1'[t] q2'[t] +
```

```
         3

                     2          4           2
   q1[t] (3 a2 q2[t]  - 2 q2[t]  - 6 q2'[t] )
>  -----------------------------------------
                         6

==================================================================

Characteristic No. : 21
------------------------------------------------------------------
      2                                                          2
{q2[t]  q1'[t] - q1[t] q2[t] q2'[t], -(q1[t] q2[t] q1'[t]) + q1[t]  q2'[t]}

Parameter: {a1, a2, G1, G2} = {a2, a2, 1, 1}

------------------------------------------------------------------
         2       2                                    2       2
    q2[t]  q1'[t]                                q1[t]  q2'[t]
I = -------------- - q1[t] q2[t] q1'[t] q2'[t] + --------------
          2                                            2

==================================================================

Characteristic No. : 22
------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a2, a2, 1, 1}

------------------------------------------------------------------
             2        4                 2       2        4         2         2
    a2 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]    q2[t]    q1'[t]    q2'[t]
I = --------- - ------ + --------------------- - ------ + ------- + -------
        4         2                4                2         2         2

==================================================================

Characteristic No. : 23
------------------------------------------------------------------
      2
{q2[t]  q1'[t] - q1[t] q2[t] q2'[t],

                                  -3 a2        2
>  -(q1[t] q2[t] q1'[t]) + (----- + q1[t] ) q2[t]}
                                    2

Parameter: {a1, a2, G1, G2} = {4 a2, a2, 1, 1}

------------------------------------------------------------------
         2      2             4        2       2
    -3 a2  q2[t]    3 a2 q2[t]    q2[t]  q1'[t]
I = ------------ + ----------- + -------------- -
          8             4              2
```

```
                                                2
                                     3 a2 q2'[t]
>    q1[t] q2[t] q1'[t] q2'[t] - ------------ +
                                          4

          2            2            2
     q1[t]  (3 a2 q2[t]  + 2 q2'[t] )
>    --------------------------------
                    4

  ====================================================================

  Characteristic No. : 24
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

  Parameter: {a1, a2, G1, G2} = {4 a2, a2, 1, 1}

  --------------------------------------------------------------------
                        4                  2       2        4         2         2
               2   q1[t]    (a2 - 4 q1[t] ) q2[t]    q2[t]    q1'[t]    q2'[t]
  I = a2 q1[t]   - ------ + ---------------------- - ------ + ------- + -------
                     2                4                2        2         2

  ====================================================================

  Characteristic No. : 25
  --------------------------------------------------------------------
        2
  {q2[t]  q1'[t] - q1[t] q2[t] q2'[t],

                                     3 a2        2
>    -(q1[t] q2[t] q1'[t]) + (---- + q1[t] ) q2'[t]}
                                       8

                                       a2
  Parameter: {a1, a2, G1, G2} = {--, a2, 1, 1}
                                       4

  --------------------------------------------------------------------
            2      2             4        2       2
      3 a2  q2[t]    3 a2 q2[t]     q2[t]  q1'[t]
  I = ------------ - ----------- + -------------- -
           32            16              2

                                                2
                                     3 a2 q2'[t]
>    q1[t] q2[t] q1'[t] q2'[t] + ------------ +
                                          16

          2             2            2
     q1[t]  (-3 a2 q2[t]  + 8 q2'[t] )
>    ---------------------------------
                    16
```

==

Characteristic No. : 26

{q1'[t], q2'[t]}

Parameter: $\{a1, a2, G1, G2\} = \{\frac{a2}{4}, a2, 1, 1\}$

$$I = \frac{a2\, q1[t]^2}{16} - \frac{q1[t]^4}{2} + \frac{(a2 - 4\, q1[t]^2)\, q2[t]^2}{4} - \frac{q2[t]^4}{2} + \frac{q1'[t]^2}{2} + \frac{q2'[t]^2}{2}$$

==

Characteristic No. : 27

{q1'[t], q2'[t]}

Parameter: $\{a1, a2, G1, G2\} = \{0, 0, \frac{1}{6}, G2\}$

$$I = \frac{-q1[t]^4}{12} - q1[t]^2\, q2[t]^2 - \frac{G2\, q2[t]^4}{2} + \frac{q1'[t]^2}{2} + \frac{q2'[t]^2}{2}$$

==

Characteristic No. : 28

{q1'[t], q2'[t]}

Parameter: $\{a1, a2, G1, G2\} = \{0, 0, \frac{1}{3}, G2\}$

$$I = \frac{-q1[t]^4}{6} - q1[t]^2\, q2[t]^2 - \frac{G2\, q2[t]^4}{2} + \frac{q1'[t]^2}{2} + \frac{q2'[t]^2}{2}$$

==

Characteristic No. : 29

{q1'[t], q2'[t]}

Parameter: $\{a1, a2, G1, G2\} = \{0, 0, 1, G2\}$

```
--------------------------------------------------------------------
          4                            4         2         2
    -q1[t]         2      2   G2 q2[t]    q1'[t]    q2'[t]
I = ------- - q1[t]  q2[t]  - --------- + ------- + -------
       2                          2          2         2

====================================================================

Characteristic No. : 30
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                     8
Parameter: {a1, a2, G1, G2} = {0, 0, -, G2}
                                     3

--------------------------------------------------------------------
            4                            4         2         2
    -4 q1[t]         2      2   G2 q2[t]    q1'[t]    q2'[t]
I = --------- - q1[t]  q2[t]  - --------- + ------- + -------
        3                           2          2         2

====================================================================

Characteristic No. : 31
--------------------------------------------------------------------
                 q1[t] q2'[t]  -(q1[t] q1'[t])
{q2[t] q1'[t] - ------------, ---------------}
                      2              2

                                     1  8
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                     6  3

--------------------------------------------------------------------
         4              2      3                2
    q1[t]  q2[t]   q1[t]  q2[t]    q2[t] q1'[t]    q1[t] q1'[t] q2'[t]
I = ------------ + ------------- + ------------- - -------------------
         6               3               2                  2

====================================================================

Characteristic No. : 32
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                     1  8
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                     6  3

--------------------------------------------------------------------
          4                           4         2         2
    -q1[t]         2      2   4 q2[t]    q1'[t]    q2'[t]
I = ------- - q1[t]  q2[t]  - -------- + ------- + -------
```

```
     12                          3          2         2

====================================================================

Characteristic No. : 33
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                    1  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                    3  3

--------------------------------------------------------------------
           4                         4         2         2
    -q1[t]          2      2   q2[t]    q1'[t]    q2'[t]
I = ------- - q1[t]  q2[t]  - ------ + ------- + -------
       6                        6         2         2

====================================================================

Characteristic No. : 34
--------------------------------------------------------------------
{q2'[t], q1'[t]}

                                    1  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                    3  3

--------------------------------------------------------------------
             3                       3
    -2 q1[t]  q2[t]   2 q1[t] q2[t]
I = --------------- - -------------- + q1'[t] q2'[t]
           3                 3

====================================================================

Characteristic No. : 35
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                    8  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                    3  6

--------------------------------------------------------------------
             4                         4         2         2
    -4 q1[t]          2      2   q2[t]    q1'[t]    q2'[t]
I = --------- - q1[t]  q2[t]  - ------ + ------- + -------
        3                         12        2         2

====================================================================

Characteristic No. : 36
--------------------------------------------------------------------
{q2[t] q2'[t], q2[t] q1'[t] - 2 q1[t] q2'[t]}
```

```
                                          8  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                          3  6

-------------------------------------------------------------------
           3      2                                      4           2
    -2 q1[t]  q2[t]                          q1[t] (q2[t]  + 3 q2'[t] )
I = ---------------- + q2[t] q1'[t] q2'[t] - --------------------------
           3                                             3

===================================================================

Characteristic No. : 37
-------------------------------------------------------------------
      2                                                            2
{q2[t]  q1'[t] - q1[t] q2[t] q2'[t], -(q1[t] q2[t] q1'[t]) + q1[t]  q2'[t]}

Parameter: {a1, a2, G1, G2} = {0, 0, 1, 1}

-------------------------------------------------------------------
         2       2                                     2       2
    q2[t]  q1'[t]                                 q1[t]  q2'[t]
I = -------------- - q1[t] q2[t] q1'[t] q2'[t] + --------------
          2                                            2

===================================================================

Characteristic No. : 38
-------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {0, 0, 1, 1}

-------------------------------------------------------------------
         4                          4         2         2
    -q1[t]         2      2    q2[t]    q1'[t]    q2'[t]
I = ------- - q1[t]  q2[t]  - ------ + ------- + -------
       2                        2         2         2

===================================================================

**** The characteristics are stored in FinalResults. ****
```

It is obvious from these results that `LieBaecklund[]` finds all the parameter combinations under which a given system of equation is integrable. The input file for the calculation contains the information

```
(*******************************************************************
*                                                                  *
* --- quartic oscillators ---                                      *
*                                                                  *
```

```
***********************************************************************)
(* --- independent variables --- *)
IndepVar={t};
(* --- dependent variables --- *)
DependVar={q1,q2};

(* --- equations  --- *)
EqList =
      {D[q1[t],{t,2}] + a1*q1[t]/2 - 2*G1*q1[t]^3 - 2*q1[t]*q2[t]^2,
       D[q2[t],{t,2}] + a2*q2[t]/2 - 2*G2*q2[t]^3 - 2*q2[t]*q1[t]^2};

(* --- parameter list --- *)
parameteR = {a1,a2,G1,G2};

(* --- substitutions --- *)
SubsList = {D[q1[t],{t,2}],D[q2[t],{t,2}]};

(* --- Characteristics --- *)
ListChar = {D[q1[t],t],D[q2[t],t]};
```

The call of `LieBaecklund[]` with the option `AnsatzPoly->{4,4}` will give all the integrals known for the quartic oscillator model.

```
 Date :  {1993, 3, 25, 11, 9, 4}
 Loading data from  4oscill.lbd


  -------------------------------------------------------------------
  Equation No. 1

  a1 q1[t]             3                2
  -------- - 2 G1 q1[t]  - 2 q1[t] q2[t]  + q1''[t] = 0
     2

  Equation No. 2

  a2 q2[t]          2                   3
  -------- - 2 q1[t]  q2[t] - 2 G2 q2[t]  + q2''[t] = 0
     2

  -------------------------------------------------------------------
LieBaecklundSymmetry`Private`derivativeOrder
Warnings -> TrueDerivativeOrder -> 1CharExpression -> {}AnsatzPoly -> {4, 4}

>   VariationalCheck -> False

  Substitution No. 1

                                     3               2
             -(a1 q1[t] - 4 G1 q1[t]  - 4 q1[t] q2[t] )
  q1''[t] -> ------------------------------------------
                                  2
```

```
Substitution No. 2

                                   2                   3
           -(a2 q2[t] - 4 q1[t]  q2[t] - 4 G2 q2[t] )
q2''[t] -> -----------------------------------------
                                   2

Substitution No. 3

  (3)                                    2                 2
q1    [t] -> -(a1 q1'[t] - 12 G1 q1[t]  q1'[t] - 4 q2[t]  q1'[t] -

>        8 q1[t] q2[t] q2'[t]) / 2

Substitution No. 4

  (3)                                                        2
q2    [t] -> -(-8 q1[t] q2[t] q1'[t] + a2 q2'[t] - 4 q1[t]  q2'[t] -

                          2
>        12 G2 q2[t]  q2'[t]) / 2

-------------------------------------------------------------------
 Determinig the Substitutions                      0.25 Second
 Creating the Characteristics                      0.0333333 Second
 Calculate generalized prolongations               0.45 Second
 Separate numerator and denominator                0.6 Second
 Substitute the higher order derivatives           0.433333 Second
 Determine the coefficients of the prolongation 6.41667 Second
 Entering simplification
 The number of equations before simplification
 *** N_before: 2


 The number of equations after simplification
 *** N_after: 2

 Simplification finished                           0.566667 Second
 Make an Ansatz for ODE's of second order
   create an Ansatz :                              3.53333 Second
   substitute the Ansatz :                         34.8667 Second
   resulting equations from the Ansatz :           723.483 Second
   calling Solve with 106 equations and          50 variables
   determine Coefficients of the Ansatz :          4150.72 Second

 Check variational symmetry
   Variationalcheck finished                       73.15 Second


Characteristic No. : 1
-------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a1, a2, G1, G2}
```

```
-------------------------------------------------------------------
                2           4                 2       2           4         2
    a1 q1[t]    G1 q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = --------- - --------- + ---------------------- - --------- + ------- +
        4           2                 4                  2          2

          2
    q2'[t]
>   -------
       2

===================================================================

Characteristic No. : 2
-------------------------------------------------------------------
              2                       2
  -3 a1 q1[t]         4   3 a1 q2[t]           2      2         4
{(------------ + q1[t]  - ----------- + 6 q1[t]  q2[t]  + q2[t] ) q1'[t] -
       2                       2

               3    3 a1 q1[t] q2[t]          3                  3
>     3 q1'[t]  + (---------------- - 2 q1[t]  q2[t] - 2 q1[t] q2[t] ) q2'[t]\
                          2

        3 a1 q1[t] q2[t]          3                  3
>     , (---------------- - 2 q1[t]  q2[t] - 2 q1[t] q2[t] ) q1'[t] +
               2

                 2                       2
      -3 a1 q1[t]         4   3 a1 q2[t]           2      2         4
>     (------------ + q1[t]  - ----------- + 6 q1[t]  q2[t]  + q2[t] )
           2                       2

                    3
>     q2'[t] - 3 q2'[t] }

                                        1  1
Parameter: {a1, a2, G1, G2} = {a1, a1, -, -}
                                        3  3

-------------------------------------------------------------------
        2      4           6        8                    2        6
    -3 a1  q1[t]    a1 q1[t]    q1[t]    (3 a1 - 4 q1[t] ) q2[t]
I = ------------- + --------- - ------ + ------------------------ -
         16             4         12                12

         8        2                 2         2          4
    q2[t]    q1[t]  (-3 a1 + 2 q1[t] ) q1'[t]    3 q1'[t]
>   ------ + -------------------------------- - --------- +
      12                    4                       4

         4       2                2          4           2
    q2[t]  (-9 a1  + 36 a1 q1[t]  - 88 q1[t]  + 24 q1'[t] )
>   ---------------------------------------------------- +
```

```
                               48

          2            4          6              2           2       2
>    q2[t]  (9 a1 q1[t]  - 4 q1[t]  - 9 a1 q1'[t]  + 36 q1[t]  q1'[t] )
>    ------------------------------------------------------------------ +
                                     12

                                2          2
>    q1[t] q2[t] (3 a1 - 4 q1[t]  - 4 q2[t] ) q1'[t] q2'[t]
>    ------------------------------------------------------ +
                               2

                  2          4             2           2      2          4
>    ((-3 a1 q1[t]  + 2 q1[t]  - 3 a1 q2[t]  + 12 q1[t]  q2[t]  + 2 q2[t] )

                               4
              2        3 q2'[t]
>       q2'[t] ) / 4 - ---------
                           4

  ===================================================================

  Characteristic No. : 3
  -------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                         1  1
  Parameter: {a1, a2, G1, G2} = {a1, a1, -, -}
                                         3  3

  -------------------------------------------------------------------
              2        4                2       2       4         2         2
      a1 q1[t]    q1[t]    (a1 - 4 q1[t] ) q2[t]   q2[t]    q1'[t]    q2'[t]
  I = --------- - ------ + --------------------- - ------ + ------- + -------
          4         6                4                6         2         2

  ===================================================================

  Characteristic No. : 4
  -------------------------------------------------------------------
  {q2'[t], q1'[t]}

                                         1  1
  Parameter: {a1, a2, G1, G2} = {a1, a1, -, -}
                                         3  3

  -------------------------------------------------------------------
                           2                       3
      q1[t] (3 a1 - 4 q1[t] ) q2[t]   2 q1[t] q2[t]
  I = ----------------------------- - -------------- + q1'[t] q2'[t]
                    6                       3

  ===================================================================

  Characteristic No. : 5
```

```
--------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {0, a2, G1, G2}

--------------------------------------------------------------------
              4                  2       2           4          2          2
    -(G1 q1[t] )   (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]    q2'[t]
I = ------------ + --------------------- - --------- + ------- + -------
         2                   4                  2          2         2

====================================================================

Characteristic No. : 6
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                     a1
Parameter: {a1, a2, G1, G2} = {a1, --, G1, G2}
                                     4

--------------------------------------------------------------------
             2           4                  2       2           4          2
    a1 q1[t]    G1 q1[t]    (a1 - 16 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = --------- - --------- + ---------------------- - --------- + ------- +
        4           2                 16                  2          2

          2
   q2'[t]
>  -------
      2

====================================================================

Characteristic No. : 7
--------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a1, a1, G1, G2}

--------------------------------------------------------------------
             2           4                 2       2           4          2
    a1 q1[t]    G1 q1[t]    (a1 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = --------- - --------- + --------------------- - --------- + ------- +
        4           2                 4                  2          2

          2
   q2'[t]
>  -------
      2

====================================================================

Characteristic No. : 8
--------------------------------------------------------------------
```

```
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a1, 4 a1, G1, G2}

-------------------------------------------------------------------
            2           4              2       2           4         2
    a1 q1[t]    G1 q1[t]                                   G2 q2[t]    q1'[t]
I = --------- - --------- + (a1 - q1[t] ) q2[t]  - --------- + ------- +
        4           2                                  2          2

         2
   q2'[t]
>  -------
      2

===================================================================

Characteristic No. : 9
-------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a1, a2, 1, G2}

-------------------------------------------------------------------
            2        4                2       2          4         2
    a1 q1[t]    q1[t]    (a2 - 4 q1[t] ) q2[t]   G2 q2[t]    q1'[t]
I = --------- - ------ + --------------------- - --------- + ------- +
        4         2                4                 2          2

         2
   q2'[t]
>  -------
      2

===================================================================

Characteristic No. : 10
-------------------------------------------------------------------
       4          2      2        4                    3
{(q1[t]  + 6 q1[t]  q2[t]  + q2[t] ) q1'[t] - 3 q1'[t]  +

             3                      3
>   (-2 q1[t]  q2[t] - 2 q1[t] q2[t] ) q2'[t],

            3                      3
>  (-2 q1[t]  q2[t] - 2 q1[t] q2[t] ) q1'[t] +

          4          2      2        4                   3
>   (q1[t]  + 6 q1[t]  q2[t]  + q2[t] ) q2'[t] - 3 q2'[t] }

                                     1  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                     3  3

-------------------------------------------------------------------
```

```
           8         2       6         8        4       2           4
    -q1[t]     q1[t]  q2[t]     q2[t]     q1[t]  q1'[t]     3 q1'[t]
I = ------- - ------------- - ------ + -------------- - --------- +
      12            3            12           2              4

        4           4            2          2      2        4          2
   q2[t]  (-11 q1[t]  + 3 q1'[t] )   q1[t]  q2[t]  (-q1[t]  + 9 q1'[t] )
>  ------------------------------ + ---------------------------------- -
                 6                                  3

                        2        2
>  2 q1[t] q2[t] (q1[t]  + q2[t] ) q1'[t] q2'[t] +

         4          2      2        4        2          4
   (q1[t]  + 6 q1[t]  q2[t]  + q2[t] ) q2'[t]    3 q2'[t]
>  ---------------------------------------- - ---------
                      2                           4

====================================================================

Characteristic No. : 11
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                        1  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                        3  3

--------------------------------------------------------------------
           4                        4        2         2
    -q1[t]          2      2   q2[t]    q1'[t]    q2'[t]
I = ------- - q1[t]  q2[t]  - ------ + ------- + -------
       6                         6        2         2

====================================================================

Characteristic No. : 12
--------------------------------------------------------------------
{q2'[t], q1'[t]}

                                        1  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                        3  3

--------------------------------------------------------------------
            3                      3
    -2 q1[t]  q2[t]   2 q1[t] q2[t]
I = --------------- - -------------- + q1'[t] q2'[t]
           3                 3

====================================================================

Characteristic No. : 13
--------------------------------------------------------------------
           2                      2
```

```
      -3 a1 q1[t]         4   3 a1 q2[t]          2      2        4
  {(------------ + q1[t]  - ----------- + 6 q1[t]  q2[t]  + q2[t] ) q1'[t] -
         2                      2

             3    3 a1 q1[t] q2[t]          3                      3
>     3 q1'[t]  + (---------------- - 2 q1[t]  q2[t] - 2 q1[t] q2[t] ) q2'[t]\
                          2

        3 a1 q1[t] q2[t]          3                      3
>     , (---------------- - 2 q1[t]  q2[t] - 2 q1[t] q2[t] ) q1'[t] +
                2

                  2                      2
       -3 a1 q1[t]          4   3 a1 q2[t]          2      2        4
>     (------------ + q1[t]  - ----------- + 6 q1[t]  q2[t]  + q2[t] )
            2                      2

                    3
>      q2'[t] - 3 q2'[t] }

                                         1  1
  Parameter: {a1, a2, G1, G2} = {a1, a1, -, -}
                                         3  3

  -------------------------------------------------------------------
          2      4           6        8                     2      6
     -3 a1  q1[t]    a1 q1[t]    q1[t]    (3 a1 - 4 q1[t] ) q2[t]
  I = ------------- + --------- - ------ + ------------------------ -
          16              4         12               12

          8        2                  2        2         4
     q2[t]    q1[t]  (-3 a1 + 2 q1[t] ) q1'[t]    3 q1'[t]
>    ------ + -------------------------------- - --------- +
       12                    4                       4

          4      2              2          4          2
     q2[t]  (-9 a1  + 36 a1 q1[t]  - 88 q1[t]  + 24 q1'[t] )
>    ------------------------------------------------------ +
                               48

          2             4          6             2            2       2
     q2[t]  (9 a1 q1[t]  - 4 q1[t]  - 9 a1 q1'[t]  + 36 q1[t]  q1'[t] )
>    ----------------------------------------------------------------- +
                                     12

                                    2          2
     q1[t] q2[t] (3 a1 - 4 q1[t]  - 4 q2[t] ) q1'[t] q2'[t]
>    ------------------------------------------------------ +
                                  2

                  2          4             2           2      2          4
>    ((-3 a1 q1[t]  + 2 q1[t]  - 3 a1 q2[t]  + 12 q1[t]  q2[t]  + 2 q2[t] )

                                 4
             2        3 q2'[t]
```

```
>       q2'[t] ) / 4 - ---------
                               4

  ====================================================================

  Characteristic No. : 14
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                             1  1
  Parameter: {a1, a2, G1, G2} = {a1, a1, -, -}
                                             3  3

  --------------------------------------------------------------------
                2        4                  2       2        4         2         2
      a1 q1[t]     q1[t]     (a1 - 4 q1[t] ) q2[t]     q2[t]     q1'[t]     q2'[t]
  I = --------- - ------ + --------------------- - ------ + ------- + -------
          4          6               4                6        2          2

  ====================================================================

  Characteristic No. : 15
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

  Parameter: {a1, a2, G1, G2} = {a1, a2, 1, 1}

  --------------------------------------------------------------------
                2        4                  2       2        4         2         2
      a1 q1[t]     q1[t]     (a2 - 4 q1[t] ) q2[t]     q2[t]     q1'[t]     q2'[t]
  I = --------- - ------ + --------------------- - ------ + ------- + -------
          4          2               4                2        2          2

  ====================================================================

  Characteristic No. : 16
  --------------------------------------------------------------------
           2
  {-(q2[t]  q1'[t]) + q1[t] q2[t] q2'[t],

                                a1 - a2        2
>     q1[t] q2[t] q1'[t] + (------- - q1[t] ) q2'[t]}
                                   2

  Parameter: {a1, a2, G1, G2} = {a1, a2, 1, 1}

  --------------------------------------------------------------------
                        4
      (-a1 + a2) q2[t]
  I = ----------------- +
              4

           2            2             2              2            2
      q2[t]  (a1 a2 - a2  - 2 a1 q1[t]  + 2 a2 q1[t]  - 4 q1'[t] )
>     ------------------------------------------------------------ +
```

```
                              8

                                                 2        2
                                (a1 - a2 - 2 q1[t] ) q2'[t]
>   q1[t] q2[t] q1'[t] q2'[t] + ----------------------------
                                             4

  ====================================================================

  Characteristic No. : 17
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

  Parameter: {a1, a2, G1, G2} = {0, 0, G1, G2}

  --------------------------------------------------------------------
                   4                                4         2         2
      -(G1 q1[t] )          2      2   G2 q2[t]    q1'[t]    q2'[t]
  I = ------------ - q1[t]  q2[t]  - --------- + ------- + -------
           2                                2          2         2

  ====================================================================

  Characteristic No. : 18
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                                 1
  Parameter: {a1, a2, G1, G2} = {a1, 4 a1, -, G2}
                                                 6

  --------------------------------------------------------------------
                2        4                   2        2             4         2
      a1 q1[t]    q1[t]                                  G2 q2[t]    q1'[t]
  I = --------- - ------ + (a1 - q1[t] ) q2[t]  - --------- + ------- +
           4        12                                        2          2

             2
     q2'[t]
>    -------
       2

  ====================================================================

  Characteristic No. : 19
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                              1
  Parameter: {a1, a2, G1, G2} = {a1, a1, -, G2}
                                              3

  --------------------------------------------------------------------
                2        4                   2        2            4          2
      a1 q1[t]    q1[t]    (a1 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
```

```
I = --------- - ------ + --------------------- - --------- + ------- +
        4         6                4                  2          2

        2
   q2'[t]
>  -------
      2

===================================================================

Characteristic No. : 20
-------------------------------------------------------------------
{q1'[t], q2'[t]}

                                               1
Parameter: {a1, a2, G1, G2} = {a1, 4 a1, -, G2}
                                               3

-------------------------------------------------------------------
            2        4                   2       2           4        2
    a1 q1[t]    q1[t]                                G2 q2[t]    q1'[t]
I = --------- - ------ + (a1 - q1[t] ) q2[t]  - --------- + ------- +
        4         6                                   2          2

        2
   q2'[t]
>  -------
      2

===================================================================

Characteristic No. : 21
-------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {0, a2, 1, G2}

-------------------------------------------------------------------
          4                2       2           4         2         2
    -q1[t]    (a2 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]    q2'[t]
I = ------- + --------------------- - --------- + ------- + -------
       2                4                  2          2         2

===================================================================

Characteristic No. : 22
-------------------------------------------------------------------
{q1'[t], q2'[t]}

                                        a1
Parameter: {a1, a2, G1, G2} = {a1, --, 1, G2}
                                        4

-------------------------------------------------------------------
            2        4                  2       2           4         2
```

```
       a1 q1[t]    q1[t]    (a1 - 16 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = --------- - ------ + ----------------------- - --------- + ------- +
          4          2                 16                      2           2

             2
     q2'[t]
>    -------
        2

====================================================================

Characteristic No. : 23
--------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a1, a1, 1, G2}

--------------------------------------------------------------------
                 2         4                  2        2            4          2
       a1 q1[t]    q1[t]    (a1 - 4 q1[t] ) q2[t]    G2 q2[t]    q1'[t]
I = --------- - ------ + ---------------------- - --------- + ------- +
          4          2                 4                     2           2

             2
     q2'[t]
>    -------
        2

====================================================================

Characteristic No. : 24
--------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a1, 4 a1, 1, G2}

--------------------------------------------------------------------
                 2         4                                             4          2
       a1 q1[t]    q1[t]                   2        2   G2 q2[t]    q1'[t]
I = --------- - ------ + (a1 - q1[t] ) q2[t]   - --------- + ------- +
          4          2                                       2           2

             2
     q2'[t]
>    -------
        2

====================================================================

Characteristic No. : 25
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                   a1   8
Parameter: {a1, a2, G1, G2} = {a1, --, -, G2}
```

4 3

$$I = \frac{a1\, q1[t]^2}{4} - \frac{4\, q1[t]^4}{3} + \frac{(a1 - 16\, q1[t]^2)\, q2[t]^2}{16} - \frac{G2\, q2[t]^4}{2} + \frac{q1'[t]^2}{2} +$$

$$> \frac{q2'[t]^2}{2}$$

===

Characteristic No. : 26

{q1'[t], q2'[t]}

Parameter: $\{a1, a2, G1, G2\} = \{a1, 4\, a1, \frac{1}{6}, \frac{8}{3}\}$

$$I = \frac{a1\, q1[t]^2}{4} - \frac{q1[t]^4}{12} + (a1 - q1[t]^2)\, q2[t]^2 - \frac{4\, q2[t]^4}{3} + \frac{q1'[t]^2}{2} + \frac{q2'[t]^2}{2}$$

===

Characteristic No. : 27

{-2 q2[t] q1'[t] + q1[t] q2'[t], q1[t] q1'[t]}

Parameter: $\{a1, a2, G1, G2\} = \{a1, 4\, a1, \frac{1}{6}, \frac{8}{3}\}$

$$I = \frac{-2\, q1[t]^2\, q2[t]^3}{3} + \frac{q2[t]\,(3\, a1\, q1[t]^2 - 2\, q1[t]^4 - 6\, q1'[t]^2)}{6} +$$

$$> q1[t]\, q1'[t]\, q2'[t]$$

===

Characteristic No. : 28

{q1'[t], q2'[t]}

Parameter: $\{a1, a2, G1, G2\} = \{a1, \frac{a1}{8}, \frac{8}{8}, -\}$

```
                                     4   3  6

  -------------------------------------------------------------------
              2          4                 2       2        4        2
      a1 q1[t]    4 q1[t]    (a1 - 16 q1[t] ) q2[t]    q2[t]    q1'[t]
  I = --------- - -------- + ---------------------- - ------ + ------- +
          4          3                16                 12         2

             2
       q2'[t]
>     -------
          2

  ===================================================================

  Characteristic No. : 29
  -------------------------------------------------------------------
  {q2[t] q2'[t], q2[t] q1'[t] - 2 q1[t] q2'[t]}

                                     a1  8  1
  Parameter: {a1, a2, G1, G2} = {a1, --, -, -}
                                     4   3  6

  -------------------------------------------------------------------
                            2       2              4
      q1[t] (3 a1 - 16 q1[t] ) q2[t]    q1[t] q2[t]
  I = ------------------------------ - ------------ + q2[t] q1'[t] q2'[t] -
                    24                       3

                   2
>    q1[t] q2'[t]

  ===================================================================

  Characteristic No. : 30
  -------------------------------------------------------------------
  {q1'[t], q2'[t]}

  Parameter: {a1, a2, G1, G2} = {a1, a1, 1, 1}

  -------------------------------------------------------------------
              2        4                2       2        4         2         2
      a1 q1[t]    q1[t]    (a1 - 4 q1[t] ) q2[t]    q2[t]    q1'[t]    q2'[t]
  I = --------- - ------ + ---------------------- - ------ + ------- + -------
          4         2                4                2         2        2

  ===================================================================

  Characteristic No. : 31
  -------------------------------------------------------------------
          2                                                         2
  {-(q2[t]  q1'[t]) + q1[t] q2[t] q2'[t], q1[t] q2[t] q1'[t] - q1[t]  q2'[t]}

  Parameter: {a1, a2, G1, G2} = {a1, a1, 1, 1}
```

```
------------------------------------------------------------------
            2       2                                  2       2
    -(q2[t]  q1'[t] )                             q1[t]  q2'[t]
I = ----------------- + q1[t] q2[t] q1'[t] q2'[t] - --------------
            2                                          2

==================================================================

Characteristic No. : 32
------------------------------------------------------------------
                 2
  -3 a1 q1[t]          4          2      2                  3
{(------------ + q1[t]  + 6 q1[t]  q2[t] ) q1'[t] - 3 q1'[t]  -
       2

           3                       3                    4
>     2 q1[t]  q2[t] q2'[t], -2 q1[t]  q2[t] q1'[t] + q1[t]  q2'[t]}

                                          1  8
Parameter: {a1, a2, G1, G2} = {a1, 4 a1, -, -}
                                          3  3

------------------------------------------------------------------
          2      4            6        8        4      4
    -3 a1  q1[t]     a1 q1[t]    q1[t]    q1[t]  q2[t]
I = ------------- + --------- - ------ - ------------- +
         16              4        12           3

        2                 2        2         4
   q1[t]  (-3 a1 + 2 q1[t] ) q1'[t]    3 q1'[t]
>  -------------------------------- - --------- +
                  4                       4

        2      2            2          4            2
   q1[t]  q2[t]  (3 a1 q1[t]  - 2 q1[t]  + 18 q1'[t] )
>  -------------------------------------------------- -
                            6

                                           4       2
          3                           q1[t]  q2'[t]
>   2 q1[t]  q2[t] q1'[t] q2'[t] + --------------
                                         2

==================================================================

Characteristic No. : 33
------------------------------------------------------------------
{q1'[t], q2'[t]}

                                          1  8
Parameter: {a1, a2, G1, G2} = {a1, 4 a1, -, -}
                                          3  3

------------------------------------------------------------------
            2        4                               4         2         2
```

```
      a1 q1[t]    q1[t]                 2        2   4 q2[t]    q1'[t]    q2'[t]
I = --------- - ------ + (a1 - q1[t] ) q2[t]  - -------- + ------- + -------
        4          6                                 3          2         2

===================================================================

Characteristic No. : 34
-------------------------------------------------------------------
      4           2      2        4                   3
{(q1[t]  + 6 q1[t]  q2[t]  + q2[t] ) q1'[t] - 3 q1'[t]  +

             3                      3
>     (-2 q1[t]  q2[t] - 2 q1[t] q2[t] ) q2'[t],

            3                      3
>    (-2 q1[t]  q2[t] - 2 q1[t] q2[t] ) q1'[t] +

          4           2      2        4                   3
>     (q1[t]  + 6 q1[t]  q2[t]  + q2[t] ) q2'[t] - 3 q2'[t] }

                                     1  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                     3  3

-------------------------------------------------------------------
           8         2      6         8         4       2           4
    -q1[t]     q1[t]  q2[t]     q2[t]     q1[t]  q1'[t]    3 q1'[t]
I = ------- - ------------- - ------ + -------------- - --------- +
      12            3           12           2               4

       4           4           2          2      2         4          2
   q2[t]  (-11 q1[t]  + 3 q1'[t] )   q1[t]  q2[t]  (-q1[t]  + 9 q1'[t] )
>  ----------------------------- + ---------------------------------- -
                 6                                 3

                        2        2
>  2 q1[t] q2[t] (q1[t]  + q2[t] ) q1'[t] q2'[t] +

         4          2      2        4         2           4
   (q1[t]  + 6 q1[t]  q2[t]  + q2[t] ) q2'[t]    3 q2'[t]
>  ----------------------------------------- - ---------
                       2                            4

===================================================================

Characteristic No. : 35
-------------------------------------------------------------------
{q1'[t], q2'[t]}

                                     1  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                     3  3

-------------------------------------------------------------------
         4                        4         2         2
```

```
      -q1[t]         2      2   q2[t]    q1'[t]    q2'[t]
  I = ------- - q1[t]  q2[t]  - ------ + ------- + -------
         6                        6         2         2

  ====================================================================

  Characteristic No. : 36
  --------------------------------------------------------------------
        4                       3
  {q2[t]  q1'[t] - 2 q1[t] q2[t]  q2'[t],

                                          2
                   3           -3 a1 q2[t]           2      2        4
>    -2 q1[t] q2[t]  q1'[t] + (------------ + 6 q1[t]  q2[t]  + q2[t] )
                                    8

                        3
>      q2'[t] - 3 q2'[t] }

                                     a1  8  1
  Parameter: {a1, a2, G1, G2} = {a1, --, -, -}
                                     4   3  3

  --------------------------------------------------------------------
                      2       6        8
      (3 a1 - 16 q1[t] ) q2[t]    q2[t]
  I = ------------------------ - ------ +
                 48                12

          4       2              2            4             2
     q2[t]  (-9 a1  + 96 a1 q1[t]  - 256 q1[t]  + 384 q1'[t] )
>    ---------------------------------------------------------- -
                                768

                  3
>    2 q1[t] q2[t]  q1'[t] q2'[t] +

          2                  2          2        2           4
     q2[t]  (-3 a1 + 48 q1[t]  + 8 q2[t] ) q2'[t]    3 q2'[t]
>    --------------------------------------------- - ---------
                          16                             4

  ====================================================================

  Characteristic No. : 37
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                     a1  8  1
  Parameter: {a1, a2, G1, G2} = {a1, --, -, -}
                                     4   3  3

  --------------------------------------------------------------------
              2          4                 2       2        4         2
      a1 q1[t]    4 q1[t]    (a1 - 16 q1[t] ) q2[t]    q2[t]    q1'[t]
```

```
I = --------- - -------- + ---------------------- - ------ + ------- +
        4          3                 16                 6         2

         2
   q2'[t]
>  -------
      2

=================================================================

Characteristic No. : 38
-----------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {a1, 4 a1, 1, 1}

-----------------------------------------------------------------
             2        4                                4         2         2
    a1 q1[t]    q1[t]                   2       2   q2[t]    q1'[t]    q2'[t]
I = --------- - ------ + (a1 - q1[t] ) q2[t]  - ------ + ------- + -------
        4         2                                 2         2         2

=================================================================

Characteristic No. : 39
-----------------------------------------------------------------
        2
{-(q2[t]  q1'[t]) + q1[t] q2[t] q2'[t],

                                -3 a1        2
>  q1[t] q2[t] q1'[t] + (----- - q1[t] ) q2'[t]}
                                  2

Parameter: {a1, a2, G1, G2} = {a1, 4 a1, 1, 1}

-----------------------------------------------------------------
               4         2       2              2           2
    3 a1 q2[t]     q2[t]  (-6 a1  + 3 a1 q1[t]  - 2 q1'[t] )
I = ----------- + ---------------------------------------- +
         4                             4

                                                  2        2
                                   (3 a1 + 2 q1[t] ) q2'[t]
>  q1[t] q2[t] q1'[t] q2'[t] - -------------------------
                                             4

=================================================================

Characteristic No. : 40
-----------------------------------------------------------------
{q1'[t], q2'[t]}

                                        a1
Parameter: {a1, a2, G1, G2} = {a1, --, 1, 1}
                                        4
```

```
  --------------------------------------------------------------------
              2        4                 2       2        4         2
      a1 q1[t]    q1[t]    (a1 - 16 q1[t] ) q2[t]    q2[t]    q1'[t]
  I = --------- - ------ + ---------------------- - ------ + ------- +
          4         2                16                2         2

           2
     q2'[t]
>    -------
        2

  ====================================================================

  Characteristic No. : 41
  --------------------------------------------------------------------
          2
  {-(q2[t]  q1'[t]) + q1[t] q2[t] q2'[t],

                           3 a1        2
>    q1[t] q2[t] q1'[t] + (---- - q1[t] ) q2'[t]}
                            8

                                     a1
  Parameter: {a1, a2, G1, G2} = {a1, --, 1, 1}
                                     4

  --------------------------------------------------------------------
                 4       2      2              2            2
      -3 a1 q2[t]   q2[t]  (3 a1  - 24 a1 q1[t]  - 64 q1'[t] )
  I = ----------- + ------------------------------------------- +
           16                          128

                                                2        2
                                 (3 a1 - 8 q1[t] ) q2'[t]
>    q1[t] q2[t] q1'[t] q2'[t] + -------------------------
                                            16

  ====================================================================

  Characteristic No. : 42
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

  Parameter: {a1, a2, G1, G2} = {0, a2, 1, 1}

  --------------------------------------------------------------------
            4                2       2        4         2         2
      -q1[t]    (a2 - 4 q1[t] ) q2[t]    q2[t]    q1'[t]    q2'[t]
  I = ------- + --------------------- - ------ + ------- + -------
         2                4                2         2         2

  ====================================================================

  Characteristic No. : 43
```

```
-------------------------------------------------------------------
          2
{-(q2[t]  q1'[t]) + q1[t] q2[t] q2'[t],

                          -a2         2
>    q1[t] q2[t] q1'[t] + (--- - q1[t] ) q2'[t]}
                           2

Parameter: {a1, a2, G1, G2} = {0, a2, 1, 1}

-------------------------------------------------------------------
            4         2     2              2          2
    a2 q2[t]    q2[t]  (-a2  + 2 a2 q1[t]  - 4 q1'[t] )
I = --------- + -------------------------------------- +
        4                         8

                                                  2         2
                                   (a2 + 2 q1[t] ) q2'[t]
>    q1[t] q2[t] q1'[t] q2'[t] - -----------------------
                                             4

===================================================================

Characteristic No. : 44
-------------------------------------------------------------------
{q1'[t], q2'[t]}

                                     1
Parameter: {a1, a2, G1, G2} = {0, 0, -, G2}
                                     6

-------------------------------------------------------------------
          4                            4          2         2
    -q1[t]          2      2   G2 q2[t]    q1'[t]    q2'[t]
I = ------- - q1[t]  q2[t]  - --------- + ------- + -------
      12                          2          2         2

===================================================================

Characteristic No. : 45
-------------------------------------------------------------------
{q1'[t], q2'[t]}

                                     1
Parameter: {a1, a2, G1, G2} = {0, 0, -, G2}
                                     3

-------------------------------------------------------------------
          4                            4          2         2
    -q1[t]          2      2   G2 q2[t]    q1'[t]    q2'[t]
I = ------- - q1[t]  q2[t]  - --------- + ------- + -------
       6                          2          2         2

===================================================================
```

Characteristic No. : 46

{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {0, 0, 1, G2}

$$I = \frac{-q1[t]^4}{2} - q1[t]^2\, q2[t]^2 - \frac{G2\, q2[t]^4}{2} + \frac{q1'[t]^2}{2} + \frac{q2'[t]^2}{2}$$

===

Characteristic No. : 47

{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = $\{0, 0, \frac{8}{3}, G2\}$

$$I = \frac{-4\, q1[t]^4}{3} - q1[t]^2\, q2[t]^2 - \frac{G2\, q2[t]^4}{2} + \frac{q1'[t]^2}{2} + \frac{q2'[t]^2}{2}$$

===

Characteristic No. : 48

{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = $\{a1, 4\, a1, \frac{1}{3}, \frac{1}{3}\}$

$$I = \frac{a1\, q1[t]^2}{4} - \frac{q1[t]^4}{6} + (a1 - q1[t]^2)\, q2[t]^2 - \frac{q2[t]^4}{6} + \frac{q1'[t]^2}{2} + \frac{q2'[t]^2}{2}$$

===

Characteristic No. : 49

{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = $\{0, 0, \frac{1}{6}, \frac{8}{3}\}$

```
          4                         4         2         2
    -q1[t]         2      2   4 q2[t]    q1'[t]    q2'[t]
I = ------- - q1[t]  q2[t]  - -------- + ------- + -------
      12                          3         2         2

====================================================================

Characteristic No. : 50
--------------------------------------------------------------------
{-2 q2[t] q1'[t] + q1[t] q2'[t], q1[t] q1'[t]}

                                           1  8
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                           6  3

--------------------------------------------------------------------
            2      3               4          2
    -2 q1[t]  q2[t]    q2[t] (q1[t]  + 3 q1'[t] )
I = ---------------- - -------------------------- + q1[t] q1'[t] q2'[t]
           3                       3

====================================================================

Characteristic No. : 51
--------------------------------------------------------------------
{q1'[t], q2'[t]}

                                           8  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                           3  6

--------------------------------------------------------------------
            4                          4         2         2
    -4 q1[t]          2      2    q2[t]    q1'[t]    q2'[t]
I = --------- - q1[t]  q2[t]  - ------ + ------- + -------
        3                         12        2         2

====================================================================

Characteristic No. : 52
--------------------------------------------------------------------
{q2[t] q2'[t], q2[t] q1'[t] - 2 q1[t] q2'[t]}

                                           8  1
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                           3  6

--------------------------------------------------------------------
            3      2              4
    -2 q1[t]  q2[t]    q1[t] q2[t]                                        2
I = ---------------- - ------------ + q2[t] q1'[t] q2'[t] - q1[t] q2'[t]
           3                3

====================================================================
```

```
Characteristic No. : 53
-------------------------------------------------------------------
{q1'[t], q2'[t]}

Parameter: {a1, a2, G1, G2} = {0, 0, 1, 1}

-------------------------------------------------------------------
          4                        4         2         2
    -q1[t]         2      2   q2[t]     q1'[t]     q2'[t]
I = ------- - q1[t]  q2[t]  - ------ + ------- + -------
       2                        2         2          2

===================================================================

Characteristic No. : 54
-------------------------------------------------------------------
        2                                                         2
{-(q2[t]  q1'[t]) + q1[t] q2[t] q2'[t], q1[t] q2[t] q1'[t] - q1[t]  q2'[t]}

Parameter: {a1, a2, G1, G2} = {0, 0, 1, 1}

-------------------------------------------------------------------
           2       2                                    2       2
    -(q2[t]  q1'[t] )                              q1[t]  q2'[t]
I = ----------------- + q1[t] q2[t] q1'[t] q2'[t] - --------------
            2                                             2

===================================================================

Characteristic No. : 55
-------------------------------------------------------------------
       4          2      2                   3          3
{(q1[t]  + 6 q1[t]  q2[t] ) q1'[t] - 3 q1'[t]  - 2 q1[t]  q2[t] q2'[t],

           3                     4
>   -2 q1[t]  q2[t] q1'[t] + q1[t]  q2'[t]}

                                     1  8
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                     3  3

-------------------------------------------------------------------
          8         4      4         4       2          4
    -q1[t]     q1[t]  q2[t]     q1[t]  q1'[t]     3 q1'[t]
I = ------- - ------------- + ------------- - --------- +
      12            3               2             4

         2      2        4          2
>   q1[t]  q2[t]  (-q1[t]  + 9 q1'[t] )           3
>   ----------------------------------- - 2 q1[t]  q2[t] q1'[t] q2'[t] +
                    3

         4       2
    q1[t]  q2'[t]
>   --------------
```

```
           2

  ====================================================================

  Characteristic No. : 56
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                       1  8
  Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                       3  3

  --------------------------------------------------------------------
            4                          4       2         2
      -q1[t]         2      2   4 q2[t]    q1'[t]    q2'[t]
  I = ------- - q1[t]  q2[t]  - -------- + ------- + -------
         6                         3          2         2

  ====================================================================

  Characteristic No. : 57
  --------------------------------------------------------------------
        4                       3
  {q2[t]  q1'[t] - 2 q1[t] q2[t]  q2'[t],

                    3                  2      2        4                  3
>    -2 q1[t] q2[t]  q1'[t] + (6 q1[t]  q2[t]  + q2[t] ) q2'[t] - 3 q2'[t] }

                                       8  1
  Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                       3  3

  --------------------------------------------------------------------
             2      6         8        4          4           2
      -(q1[t]  q2[t] )   q2[t]    q2[t]  (-2 q1[t]  + 3 q1'[t] )
  I = ---------------- - ------ + ----------------------------- -
             3             12                   6

                                          2         2        2        2
                   3                 q2[t]  (6 q1[t]  + q2[t] ) q2'[t]
>    2 q1[t] q2[t]  q1'[t] q2'[t] + --------------------------------- -
                                                   2

             4
     3 q2'[t]
>    ---------
         4

  ====================================================================

  Characteristic No. : 58
  --------------------------------------------------------------------
  {q1'[t], q2'[t]}

                                       8  1
```

TABLE II
Two dimensional quartic oscillators

Parameters	Q_1	Q_2	Integral
$\alpha_1, \alpha_2, \Gamma_1, \Gamma_2$	$\dot q_1$	$\dot q_2$	$\frac{1}{2}\sum_{i=1}^{2}\{\dot q_i^2 + \frac{\alpha_i}{2}q_i^2 - \Gamma_i q_i^4\} - q_1^2 q_2^2$
$\alpha_1 \neq \alpha_2$, Γ_1, $\Gamma_2 = 1$	$\frac{1}{2}(\alpha_1 - \alpha_2)\dot q_1 + q_2(q_2\dot q_1 - q_1\dot q_2)$	$q_1(q_1\dot q_2 - q_2\dot q_1)$	$\frac{1}{2}(\alpha_1 - \alpha_2)(q_1^4 + q_1^2 q_2^2 - \frac{\alpha_1}{2}q_1^2 - \dot q_1^2) - (q_1\dot q_2 - q_2\dot q_1)^2$
$\alpha_1 = \alpha_2$, $\Gamma_1 = 1/3$, $\Gamma_2 = 1/3$	$\dot q_2$	$\dot q_1$	$\dot q_1\dot q_2 + q_1 q_2\left(\frac{\alpha_1}{2} - \frac{2}{3}(q_1^2 + q_2^2)\right)$
$4\alpha_1 = \alpha_2$, $\Gamma_1 = 1/6$, $\Gamma_2 = 8/3$	$q_1\dot q_2 - 2q_2\dot q_1$	$q_1\dot q_1$	$\dot q_1(q_1\dot q_2 - q_2\dot q_1) - \frac{2}{3}q_1^2 q_2^3 - \frac{1}{3}q_1^4 q_2 + \frac{\alpha_1}{2}q_1^2 q_2$
$\alpha_2 = 4\alpha_1$, $\Gamma_1 = 1/3$, $\Gamma_2 = 8/3$	$\frac{8}{3}q_1^3 q_2\dot q_2 - 8q_1^2 q_2^2\dot q_1 + 4\dot q_1^3 - \frac{4}{3}q_1^4\dot q_1 + 2\alpha_1 q_1^2\dot q_1$	$\frac{8}{3}q_1^3 q_2\dot q_1 - \frac{4}{3}q_1^4\dot q_2$	$\dot q_1^4 + 4\left(\frac{\alpha_1}{4} - \frac{1}{6}q_1^2 - q_2^2\right)q_1^2\dot q_1^2 + \frac{8}{3}q_1^3 q_2\dot q_1\dot q_2 + \frac{1}{9}(q_1^2 + 2q_2^2)^2 q_1^4 - \frac{2}{3}q_1^4\dot q_2^2 + \alpha_1\left(\frac{\alpha_1}{4} - \frac{1}{3}q_1^2 - \frac{2}{3}q_2^2\right)q_1^4$

```
Parameter: {a1, a2, G1, G2} = {0, 0, -, -}
                                      3  3

-------------------------------------------------------------------
           4                             4         2         2
    -4 q1[t]          2      2    q2[t]      q1'[t]     q2'[t]
I = --------- - q1[t]  q2[t]   - ------ + ------- + -------
        3                            6         2         2

===================================================================

**** The characteristics are stored in FinalResults. ****
```

8.3 Two ions in a trap

The two-ion motion confined to a Paul trap has been frequently discussed in different contexts. Experiments were carried out by Blümel et al [9] and Hoffnagle et al [10]. The experimental settings of a dynamical ion trap were first described by Paul et al [11] in the late fifties. The motion of two ions in such a trap is governed mainly by two forces, the quadrupole force $F^{(trap)}$ and the Coulomb forces $F^{(coul)}$. The quadrupole force in the center of the trap is derivable from the time dependent potential $\Phi^{(trap)} = (U_0 + V_0 \cos(\Omega t))(x^2 + y^2 - 2z^2)/(r_0^2 + 2z_0^2)$, where U_0, V_0 and Ω denote amplitudes of *dc* and *ac* voltage, and Ω is the corresponding driving frequency. r_0 and z_0 give the geometric extensions of the trap in radial and axial directions. The basic equations which we will consider here are derivable in a secular approximation from the generalized Mathieu equations for the relative coordinates. Applying Landau's fast field averaging [12], we can write down the equations of motion in cylindrical coordinates to read

$$\Delta = \begin{pmatrix} \ddot{\rho} - \frac{\nu^2}{\rho^3} + \rho - \dfrac{\rho}{(\rho^2 + \zeta^2)^{3/2}} \\ \ddot{\zeta} + \lambda^2 \zeta - \dfrac{\zeta}{(\rho^2 + \zeta^2)^{3/2}} \end{pmatrix} = 0. \tag{27}$$

λ is the ratio of secular frequencies and ν is the ratio of angular momentum ℓ_z and of the radial secular frequency. The coordinates of relative motion ρ, ζ and time τ are scaled by an appropriate length and the driving frequency. The second order equations (27) are derivable from the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{\rho}^2 + \dot{\zeta}^2) - \{\frac{1}{2}(\rho^2 + \lambda^2\zeta^2) + \frac{1}{(\rho^2 + \zeta^2)^{1/2}} + \frac{\nu^2}{2\rho^2}\}. \tag{28}$$

It is evident that equations (27) are highly nonlinear. With $\lambda = 0$, they have some resemblance to a two dimensional Kepler problem. In the language of classical mechanics, the equations describe the motion of one particle in a two-dimensional potential. The second ion in this picture is fixed at the origin of coordinates. The phase space of this Hamiltonian system is four-dimensional and thus is a potential candidate for chaotic motion. The question now is under which parameter conditions regular motion can be found.

We will consider transformations of dependent and independent variables (ρ,-$\zeta, \dot{\rho}, \dot{\zeta}, t$) to carry over solutions of (27) into solutions. We take invariance into account which is represented by equation (20). The vector field $\vec{v}$ in an evolutionary representation is here given by (18). The characteristics Q_i are functions of the dependent and independent variables $Q_i = Q_i(\rho, \zeta, \dot{\rho}, \dot{\zeta}, t)$ $(i = 1, 2)$. Expanding equation (20), we obtain a linear system of partial differential equations determining the characteristics Q_i.

Preliminary to the general procedure, we examined point symmetries of equations (27) by using the *Mathematica* program `Lie[]` [6]. From a set of 18 equations, we obtained the characteristics Q_1 and Q_2 to be

$$Q_1 = -k_2\zeta - k_1\dot{\rho} \qquad Q_2 = k_2\rho - k_1\dot{\zeta}, \tag{29}$$

where the group constants k_1 and k_2 are arbitrary. The geometrical interpretation of k_1 and k_2 correspond to translational invariance in time and rotational invariance in the (ρ, ζ)-plane, respectively. We find that the symmetry given by k_1 exists independent of the choice of parameters λ und ν. Contrary to this global existence, rotational invariance exists only for $\lambda = \pm 1$ and $\nu = 0$. It is easy to check that the point symmetries of (27) are also symmetries of the Lagrangian (28). In cases in which symmetries are also variational symmetries, Noether's theorem provides the corresponding integrals of motion. Thus, we obtain the total energy E and the angular momentum ℓ corresponding to k_1 and k_2

$$E = \frac{1}{2}(\dot{\rho}^2 + \dot{\zeta}^2) + \frac{1}{2}(\rho^2 + \lambda^2\zeta^2) + \frac{1}{(\rho^2+\zeta^2)^{1/2}} + \frac{\nu^2}{2\rho^2} \tag{30}$$

$$\ell = \rho\dot{\zeta} - \zeta\dot{\rho}. \tag{31}$$

as integrals of motion, respectively. Provided that the z-component of the angular momentum ℓ_z equals zero and that secular frequencies in axial and radial directions are identical, we find that the motion of the two ions is completely determined by the integrals E and ℓ. These two conserved quantities restrict the motion of the ions to a two-dimensional surface in the phase space. Thus, the motion of the ions is completely regular for $\lambda = \pm 1$ and $\nu = 0$.

Enlarging the symmetry class from point symmetries to generalized symmetries, we find five different parameter sets (λ, ν) for which integrals of motion exist. The generalized symmetries are provided if we use (20) in its explicit form

$$\frac{d^2}{dt^2}Q_1 + 3\nu^2\frac{1}{\rho^4}Q_1 + Q_1 - \frac{Q_1}{(\rho^2+\zeta^2)^{3/2}} + \frac{3Q_1\rho}{(\rho^2+\zeta^2)^{5/2}} + \frac{3Q_2\zeta}{(\rho^2+\zeta^2)^{5/2}} = 0 \tag{32}$$

$$\frac{d^2}{dt^2}Q_2 + \lambda^2 Q_2 - \frac{Q_2}{(\rho^2+\zeta^2)^{3/2}} + \frac{3Q_2\zeta}{(\rho^2+\zeta^2)^{5/2}} + \frac{3Q_1\rho}{(\rho^2+\zeta^2)^{5/2}} = 0. \tag{33}$$

This system of equations delivers a linear system of equations determining the unknown functions Q_1 and Q_2. Carrying out the solution, we get five cases of parameter combinations (λ, ν)

i) λ and ν arbitrary (conservation of energy),

ii) $\lambda = \pm 1$, $\nu =$ arbitrary (conservation of angular momentum),

iii) $\lambda = \pm 2$, $\nu =$ arbitrary,

iv) $\lambda = \pm 1/2$ and $\nu = 0$, and

v) $\lambda = \pm 1/2$ and $\nu =$ arbitrary.

For each of these cases we can determine a pair of characteristics Q_1 and Q_2 given in table III. Since the characteristics depend on functions which are not of polynomial type, we cannot apply `LieBaecklund[]` to all cases.

TABLE III
Characteristics of the Paul trap containing two ions

case	Q_1	Q_2
i)	$\dot{\rho}$	$\dot{\zeta}$
ii)	$\rho\zeta\dot{\zeta} - \zeta^2\dot{\rho}$	$\rho\zeta\dot{\rho} - \rho^2\dot{\zeta}$
iii)	$\dot{\zeta}\rho - 2\zeta\dot{\rho}$	$\rho\dot{\rho}$
iv)	$\zeta\dot{\zeta}$	$\zeta\dot{\rho} - 2\rho\dot{\zeta}$
v)	$2\zeta\dot{\zeta}^2\left(\zeta\dot{\rho} - \rho\dot{\zeta}\right) + \frac{1}{2}\rho\zeta^3\dot{\zeta} - \frac{2\rho\zeta\dot{\zeta}}{(\rho^2+\zeta^2)^{(1/2)}} + 2\nu^2\dot{\rho}$	$\rho\zeta^2\left(\frac{1}{2}\zeta\dot{\rho} - \rho\dot{\zeta}\right) + 4\rho^2\dot{\zeta}^3 + \frac{2\rho}{(\rho^2+\zeta^2)^{(1/2)}}\left(2\rho\dot{\zeta} - \zeta\dot{\rho}\right) + 2\zeta\dot{\rho}\dot{\zeta}\left(\zeta\dot{\rho} - 3\rho\dot{\zeta}\right) + 2\nu^2\left(2\dot{\zeta} + \frac{\zeta^2\dot{\zeta}}{\rho^2}\right).$

A comparison of the characteristics **i)** and **ii)** with those obtained from point symmetries shows that time translation and rotational invariance are also generalized symmetries. Based upon the aforegoing, we know the invariance properties of equations (27) but not the related integrals.

To calculate the corresponding integrals of motion, we use

$$Q \cdot \Delta = \frac{d}{dt} I. \tag{34}$$

I is an invariant, $Q = (Q_1, Q_2)^T$ the characteristic, and Δ the left hand side of the Euler-Lagrange equations (27). In order to demonstrate that (34) essentially contains the information to determine integrals of motion, we will consider as an example the general case where λ and ν are arbitrary. With respect to case **i)**, we find from (34)

$$\begin{pmatrix} \dot{\rho} \\ \dot{\zeta} \end{pmatrix} \begin{pmatrix} \ddot{\rho} - \frac{\nu^2}{\rho^3} + \rho - \dfrac{\rho}{(\rho^2+\zeta^2)^{3/2}} \\ \ddot{\zeta} + \lambda^2\zeta - \dfrac{\zeta}{(\rho^2+\zeta^2)^{3/2}} \end{pmatrix} = \frac{d}{dt} E = 0, \tag{35}$$

where E can be calculated by quadrature to be

$$E = \frac{1}{2}(\dot{\rho}^2 + \dot{\zeta}^2) + \frac{1}{2}(\rho^2 + \lambda^2\zeta^2) + \frac{1}{(\rho^2+\zeta^2)^{1/2}} + \frac{1}{2}\frac{\nu^2}{\rho^2}. \tag{36}$$

The definition of momenta by $p_\rho = \dot{\rho}$ and $p_\zeta = \dot{\zeta}$ shows that E is nothing but the energy in the relative coordinates of the two ions in the Paul trap. This result is more or less trivial since we started from a Lagrangian.

With respect to case ii) with $\lambda = \pm 1$ and $\nu = 0$, we already know that ℓ is an invariant of motion. The question now is what happens if the z-component of ℓ is different from zero? If we insert the characteristics into (34) for this case, we get

$$\begin{pmatrix} \zeta(\rho\dot{\zeta} - \zeta\dot{\rho}) \\ \rho(\zeta\dot{\rho} - \rho\dot{\zeta}) \end{pmatrix} \begin{pmatrix} \ddot{\rho} - \frac{\nu^2}{\rho^3} + \rho - \dfrac{\rho}{(\rho^2+\zeta^2)^{3/2}} \\ \ddot{\zeta} + \zeta - \dfrac{\zeta}{(\rho^2+\zeta^2)^{3/2}} \end{pmatrix} = \frac{d}{dt} I^{(ii)} = 0. \tag{37}$$

The corresponding integral of motion $I^{(ii)}$ reads

$$I^{(ii)} = -\left[\frac{1}{2}(\rho\dot{\zeta} - \zeta\dot{\rho})^2 + \frac{1}{2}\frac{\nu^2}{\rho^2}\zeta^2\right]. \tag{38}$$

The expression (38) contains the square of the angular momentum ℓ and a term depending on the square of ℓ_z. At this point, we derive the first new result from the generalized symmetry analysis. In (38), we have a quadratic dependence in ζ/ρ generalizing the invariant ℓ known from point analysis.

With respect to case iii) with $\lambda = \pm 2$ and ν arbitrary, we can determine the second integral of motion in the same fashion to be

$$I^{(iii)} = \rho\dot{\zeta}\dot{\rho} - \zeta\dot{\rho}^2 + \rho^2\zeta - \frac{\zeta}{(\rho^2+\zeta^2)^{1/2}} - \frac{\nu^2\zeta}{\rho^2}. \tag{39}$$

Since ν is arbitrary in the calculation of the characteristics, we can choose $\nu = 0$ in order to find correspondence with results published by Blümel et al [8]

$$I^{(iii)}\Big|_{\nu=0} = \rho\dot{\zeta}\dot{\rho} - \zeta\dot{\rho}^2 + \rho^2\zeta - \frac{\zeta}{(\rho^2+\zeta^2)^{1/2}}. \tag{40}$$

With respect to cases iv) and v), a different situation appears. Here, the symmetry analysis allows the combinations $\lambda = \pm 1/2$, $\nu = 0$, and $\lambda = \pm 1/2$, $\nu =$ arbitrary. The generalized symmetry for $\nu = 0$ gives the second integral of motion in the form

$$I^{(iv)} = \dot{\zeta}\dot{\rho}\zeta - \rho\dot{\zeta}^2 + \frac{1}{4}\zeta^2\rho - \frac{\rho}{(\rho^2+\zeta^2)^{1/2}}. \tag{41}$$

This is the second case where integrals of motion given in [8] coincide with our results. Contrary to the relations in [8], we find from the generalized symmetry v) with $\nu \neq 0$ and $\lambda = \pm 1/2$ the following representation of the second integral

$$\begin{aligned} I^{(v)} \;=\; & \frac{\nu^4}{\rho^2} + \nu^2\rho^2 + \frac{\nu^2\zeta^2}{2} + \frac{\rho^2\zeta^4}{16} + \frac{\rho^2}{\rho^2+\zeta^2} + \frac{2\,\nu^2}{\sqrt{\rho^2+\zeta^2}} \\ & - \frac{\rho^2\zeta^2}{2\sqrt{\rho^2+\zeta^2}} + \nu^2\dot{\rho}^2 + \frac{\rho\,\zeta^3\dot{\rho}\,\dot{\zeta}}{2} - \frac{2\,\rho\,\zeta\,\dot{\rho}\,\dot{\zeta}}{\sqrt{\rho^2+\zeta^2}} \\ & + 2\,\nu^2\dot{\zeta}^2 + \frac{\nu^2\zeta^2\dot{\zeta}^2}{\rho^2} - \frac{\rho^2\zeta^2\dot{\zeta}^2}{2} + \frac{2\,\rho^2\dot{\zeta}^2}{\sqrt{\rho^2+\zeta^2}} + \zeta^2\dot{\rho}^2\dot{\zeta}^2 \\ & - 2\,\rho\,\zeta\,\dot{\rho}\,\dot{\zeta}^3 + \rho^2\dot{\zeta}^4. \end{aligned} \tag{42}$$

Thus, we succeeded in an analytic construction of the second integral of motion for this case.

However, we note that in cases **ii)** - **v)**, there exists beside of the given second integrals the integral of energy. In all cases for which generalized symmetries are known, we find that the two ions are confined to a two-dimensional surface in phase space and are thus integrable. The reason for the integrability is the fact that the ratios of the two secular frequencies have specific values. With respect to case **ii)**, both frequencies are equal; in case **iii)**, the axial frequency is twice as large as the radial frequency ($\lambda = \pm 2$); and with respect to $\lambda = \pm 1/2$, the axial frequency is one half of the radial frequency. We note that the characteristics in cases **iii)** and **iv)** are well known from the classical Kepler problem corresponding to the Runge-Lenz vector. The difference between the classical Kepler problem with its $1/r$ dependence on the potential and the Paul trap is an additional harmonic part in the potential and its two-dimensionality. However, the symmetry of the Runge-Lenz vector of the Kepler problem is identical to the symmetry of the Paul trap. What has changed are the corresponding integrals of motion for the Paul trap.

Similar calculations for the Hénon Heiles model and a quartic anharmonic oscillator can be easily done. Using the *Mathematica* program `LieBaecklund[]` the integrals of motion and the special parameter combinations are derivable in a straightforward way. The results of such calculations are given in the Appendix.

9 Concluding remarks

We demonstrated the application of the generalized symmetry method (Lie–Bäcklund method) on some model equations. We find that the application of this method will give us a straightforward way to determine the related integrals of motion and to decide the question of integrability. For the Hénon–Heiles model, the quartic anharmonic oszillators, and the two ions in a Paul trap, we calculated integrals of motion by applying a generalization of Lie's method. The calculation for all models was supported by the *Mathematica* programs `Lie[]` and `LieBaecklund[]`. These programs allow us to find the simplified determining equations of the characteristics and make an ansatz of polynomial form in case of the `LieBaecklund[]` program for second order ODEs. However, this ansatz of polynomials in the coordinates and derivatives does not give us the most general solutions for the characteristics but instead provides us with a form which is managable on a computer. The Lie group approach in its generalized form is particularly effective since it furnishes both the generalized Lie symmetries and all their invariants. For the models under discussion, it was possible to identify all generalized symmetries with variational symmetries and thus, the integrals of motion are available. The relations among the parameters follow as side conditions from a linear system of determining equations. By using the polynomial ansatz for the characteristics, we have an effective scheme to determine the symmetries. The corresponding parameter combinations are consistent with those derived from Painlevé tests. The advantage of the method under consideration is that it furnishes the parameters and the integrals of motion in a constructive way.

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10 Program Listing

File backlund.m

```
(*:Mathematica Version: 2.0 *)

(*:Name: Calculus`LieBaecklundSymmetry` *)

(*:Title: LieBaecklund Symmetries *)

(*:Author: Gerd Baumann
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           D-7900 Ulm
           Germany

      phone:    +731-502-2996
      fax:      +731-502-3003
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*)

(*:Summary:

Implements the calculation of Lie-Baecklund symmetries for systems of partial
```

```
and ordinary differential equations

*)

(*:Keywords:

Lie;
Baecklund;
Lie-Baecklund symmetries;
symmetries;
transformations;
differential equations;
partial differential equations;
ordinary differential equations;

*)

(*:Requirements: none. *)

(*:Sources:
      P. Olver, Applications of Lie Groups to Differential Equations
               Springer, New York, 1986
      G.W. Bluman and S. Kumei, Symmetries and Differential Equations
               Springer, New York, 1989
*)

(*:History:
*)

(*:Discussion:
      The purpos of this program is to derive the determining equations for
      Lie-Baecklund symmetries. The determining equations are derived in a form
      simplified as far as possible. The program allows an automatic solution
      of the determining equations for second order ODEs by using a polynomial
      ansatz in the dependent variables and derivatives. If the
      characteristics for second order ODes are calculated the program
      determines the corresponding integrals of motion.

*)

BeginPackage["LieBaecklundSymmetry`"]

Clear[Variab,Characteristics,DList,Prolongation,LieBaecklund,Substit,
      Searchcof,Searchf1,Searchf2,PowerT,Simplifyeqns,CancelPreFactor,
      Elimin,LTest,SubstRule,Substitution,Baecklund,AnsatzPolynom,
      Ansatzcof,DetCoeff,adjointFrechet,Frechet,CharacteristicTwo,
      Monomials,PrintCharacteristic,PrintEquations,PrintIntegrals,
      ConstantCases]

Print["  +======================================================================= "];
Print["  |  "];
Print["  |              LieBaecklund Version 1.0 for Mathematica 2.0"];
Print["  |                       Copyright 1993 G. Baumann"];
Print["  |  "];
Print["  +-----------------------------------------------------------------------"]
```

```
Print["  | **** The functions defined in the package LieBaecklundSymmetry are"]
Print["  +--------------------------------------------------------------------"]
Print["  | "]
Print["  | LieBaecklund      : calculates the determining equations in batch mode "]
Print["  | Baecklund         : calculates the determining equations intaractively "]
Print["  | CharacteristicTwo : calculates the commutator of two symmetriese "]
Print["  |  "]
Print["  | To get more information type ?name"]
Print["  |  "]
Print["  +==================================================================== "];

Baecklund::usage="Baecklund[expr_List, subst_List,
          independent_List, dependent_List,order_,options___]
          calculates the determining
          equations for generalized symmetries of a system of partial or
          ordinary differential equations. \n \n
          \t EXPR \t:   contains a list of equations \n
          \t SUBST \t:  gives the terms to be substituted \n
          \t INDEPENDENT \t: list of independent variables \n
          \t DEPENDENT \t: a list of dependent variables \n
          \t ORDER \t: order of derivatives contained in the characteristics
          \n
          \t OPTIONS \t: options like CharExpression and Warnings
          \n \n
\n
Example:
\n Baecklund[{D[u[x,t],t]-D[u[x,t],{x,2}]},{D[u[x,t],t]],{x,t},{u},2]
\n \n
LieBaecklund calculates the generalized prolongation of the given equations
and performes substitutin of the higher derivatives of SUBST. A list of the
determining equations is returned. The elements of this list are the
coefficients comming from derivatives greater than ORDER.
\n \n
The following options are used in the calculation
\n \n
Warnings -> True  : give warnings on cancelation of factors
\n
CharExpression -> {} : List contains expressions for the characteristics
\n
AnsatzPoly->{0,0} : make an Anstz for the characteristics in case of
ODE's of second order
\n
VariationalCheck->False : check the given characteristics to be a variational
symmetry if True calculate the corresponding integral
\n
\n
The given Values of the options are default values.
"


LieBaecklund::usage="LieBaecklund[name_String,options___] reads the data from
file name and calls LieBaecklund[] to do the calculation
\n \n
There are the following options to influence the calculation
\n \n
```

```
Warnings -> True  : give warnings on cancelation of factors
\n
DerivativeOrder -> 1  : the highest order of derivatives used in the
characteristics
\n
CharExpression -> {} : List contains expressions for the characteristics
\n
AnsatzPoly->{0,0} : make an Anstz for the characteristics in case of
ODE's of second order
\n
VariationalCheck->False : check the given characteristics to be a variational
symmetry if true calculate the corresponding integral
\n
\n
The given Values of the options are default values.
\n
\n
\n
(* ---    Diffusion Equation      --- *)      \n
(*-----------------------------------*)      \n
(* --- independent variables --- *)          \n
IndepVar={x,t};                              \n
(* --- dependent variables --- *)            \n
DependVar={u};                               \n
(* --- equations  --- *)                     \n
EqList = {D[u[x,t],t] - D[u[x,t],{x,2}]}     \n
(* --- parameters --- *)                     \n
parameteR = {}                               \n
(* --- substitutions --- *)                  \n
SubsList = {D[u[x,t],t]};                    \n
(* --- lists of the characteristics --- *)   \n
ListChar = {D[u[x,t],x]};                    \n
\n
\n
"

CharacteristicTwo::usage="CharacteristicTwo[Qin_List, Rin_List, indVar_List, dependVar_List]
calculates the characteristic of two vector fields v_Q and v_R by means of
proposition 5.15 of Olver. S = [v_Q,v_R]. This function is bilinear with
respect to the first argument."


(* --- global variables --- *)

FinalResults::usage="List containing the results of the calculation"
EqList::usage="Contains the equations of motion in a list"
SubsList::usage="Contains the substititions in a list"
IndepVar::usage="List of independent variables"
DependVar::usage="List of dependent variables"
parameteR::usage="List containing the parameters of the equations"

(* --- global options --- *)
Warnings::usage="Display warnings if set to True"
DerivativeOrder::usage="Order of derivatives used in the characteristics"
CharExpression::usage="Contains the expressions for the characteristics"
```

```
AnsatzPoly::usage="A list for the ansatz used for the charcteristics"
VariationalCheck::usage"Check variational symmetry"

Begin["`Private`"]

(* --- Creates the dependencies of the characteristics --- *)

Variab[independent_List, dependent_List, order_, exclude_]:=Block[
      {varlist = {},
      dvarlist = {},
      derivlist = {},dlist,independdiff},

(* --- Exclude one independent variable --- *)
      independdiff = DeleteCases[independent,exclude];

(* --- Create the dependencies --- *)
      AppendTo[varlist,independent];

      Do[AppendTo[dvarlist,dependent[[i]][Delete[independent,0]]],
      {i,1,Length[dependent]}];

      dlist = dvarlist;

(* --- All derivatives up to order ORDER --- *)
            Do[dlist = Outer[D,dlist,independdiff];
               AppendTo[derivlist,dlist],
            {i,1,order}];

      derivlist = Flatten[derivlist];
      derivlist = Union[derivlist];
      AppendTo[varlist,{dvarlist,derivlist}];
      varlist = Flatten[varlist]
      ]

(* --- Creates the characteristics --- *)

Characteristics[independent_List, dependent_List, order_, exclude_]:=Block[
      {character,variabel},
      character = {};
      variabel = Variab[independent, dependent, order, exclude];
      Do[AppendTo[character,Unique["QC"][Delete[variabel,0]]],
      {i,1,Length[dependent]}];
      character
      ]

(* --- Serves to generate a nested list --- *)

DList[y_,x_]:={y,x}


(* --- Determines maximal order of differentiation --- *)

MaxD[expr_,dependVar_]:=Block[{lengthExpr,level,case,order,x,y},
             lengthExpr = Depth[expr];
             level = Level[expr,lengthExpr];
```

```
            level = AppendTo[level,expr];
            case = Cases[level,Derivative[x___][dependVar][y___]->{x}];
            case = Apply[Plus,case,2];
            If[case == {},order = 0,order = Max[case]]
            ]

(* --- Generalized prolongation of a vectorfield --- *)

(* --- Applies the k-th prolongation of the vector field  --- *)
(* --- v to a function depending on the independent and   --- *)
(* --- dependent variables.                               --- *)
(* --- Formulas are identical with (5.6) of               --- *)
(* --- P. Olver (1986) p. 297                             --- *)

Prolongation[delta_, indVar_List, dependVar_List, characteristics_List,
             order_] := Block[
{lind,indepVar,lenin,lendep,dlist,d1list,Prlist,denslist,
 Jcase,Jlist,erg,Phij,charac},

      DeriList[x_,y_]:={x,y};

(* --- Length of lists for independent and dependent variables --- *)
      lind = indVar;
      indepVar = Delete[lind,0];
      lenin = Length[lind];
      lendep = Length[dependVar];
      erg = 0;
      charac = characteristics;

(* --- Calculation of the prolongation; sum over dependent variables --- *)
      Do[
      dlist = {dependVar[[ndep]][indepVar]};
      d1list = {delta};

(* --- Loop over orders --- *)
      Do[
      Prlist = {};
(* --- Determine all derivatives of the dependent variables up   --- *)
(* --- to order n and cancel all multiple terms                  --- *)
            dlist = Outer[D,dlist,{indepVar}];
            dlist = Union[Flatten[dlist]];

(* --- Calculate the derivatives with respect to the gradients --- *)
      denslist = Flatten[Outer[D,d1list,dlist]];
         Do[If[ Not[ denslist[[i]] === 0],
           Jcase = Cases[{dlist[[i]]},
                         Derivative[x___][dependVar[[ndep]]][y___]->{{y},{x}}];
             Jlist = {};
             AppendTo[Jlist, Thread[ DeriList[Jcase[[1,1]],Jcase[[1,2]] ] ]];
             Jlist = Flatten[Jlist,1];
             Jlist = Delete[Jlist,0];

(* --- Equation (5.6) of Olver --- *)
           Phij = D[charac[[ndep]],Jlist];
           AppendTo[Prlist,Phij*D[ delta ,
```

```
                                D[dependVar[[ndep]][indepVar],Jlist ] ]]
            ],
          {i,1,Length[denslist]}];
      erg = erg + Apply[Plus,Prlist],
      {n,1,order}],
      {ndep,1,lendep}];
      erg = erg + Sum[charac[[ndep]]*D[delta,dependVar[[ndep]][indepVar]],
                  {ndep,1,lendep}];
      erg = Expand[erg]
      ];

(* --- Evolutionary vectorfield built up of two vectorfields --- *)
(*     Proposition 5.15 p. 308 of Olver                          *)

CharacteristicTwo[Qin_List, Rin_List, indVar_List, dependVar_List]:=
    Block[{lenq,lenr,lendep,sumS},
    lenq = Length[Qin];
    lenr = Length[Rin];
    lendep = Length[dependVar];
    sumS = {};
    ordeR ={};

    If[(lenq == lendep && lenr == lendep),

(* --- Determination of maximal differentiation order --- *)
       Do[AppendTo[ordeR,MaxD[ Qin,dependVar[[i]] ] ],
       {i,1,Length[dependVar]}];
       Do[AppendTo[ordeR,MaxD[ Rin,dependVar[[i]] ] ],
       {i,1,Length[dependVar]}];
       ordeR = Flatten[ordeR];
       order = Max[ordeR];

(* --- calculate the Lie bracket --- *)
       Do[
          AppendTo[sumS,Expand[
                Prolongation[Rin[[i]],indVar,dependVar,Qin,order] -
                Prolongation[Qin[[i]],indVar,dependVar,Rin,order]] ],
       {i,1,lenq}],
    Print["  #### Length missmatch in the characteristics!"]
    ];
    sumS
    ]


(* --- Biliniarity of the Lie bracket in characteristics form --- *)

CharacteristicTwo[c1_.*Qin1_List+c2_.*Qin2_List,
                Rin_List, indVar_List, dependVar_List]:=
c1*CharacteristicTwo[Qin1,Rin,indVar,dependVar] +
c2*CharacteristicTwo[Qin2,Rin,indVar,dependVar]



(* --- adjoint Frechet derivative --- *)
(* --- Formulas are identical with (5.48) of             --- *)
```

```
(* --- P. Olver (1986) p. 323                                --- *)
(* ---
      order : determines the highest order of derivatives in the
              characteristics
                                                               --- *)


adjointFrechet[delta_List, indVar_List, dependVar_List, characteristics_List,
              order_] := Block[
{lind,indepVar,lenin,lendep,dlist,d1list,Prlist,denslist,
 Jcase,Jlist,Jterm,erg,Phij,Phisum,charac},

      DeriList[x_,y_]:={x,y};

(* --- Length of lists for independent and dependent variables --- *)
      lind = indVar;
      indepVar = Delete[lind,0];
      lenin = Length[lind];
      lendep = Length[dependVar];
      erg = 0;

      charac = characteristics;

(* --- Calculation of the adjoint Frechet derivative;
                         sum over dependent variables --- *)
      Do[
      dlist = {dependVar[[ndep]][indepVar]};
      d1list = charac;

(* --- Loop over orders --- *)
      Do[
      Prlist = {};
(* --- Determine all derivatives of the dependent variables up   --- *)
(* --- to order n and cancel all multiple terms                  --- *)
            dlist = Outer[D,dlist,{indepVar}];
            dlist = Union[Flatten[dlist]];

(* --- Calculate the derivatives with respect to the gradients --- *)
      denslist = Flatten[Outer[D,d1list,dlist]];
         Do[If[ Not[ denslist[[i]] === 0],
           Jcase = Cases[dlist,
                         Derivative[x___][dependVar[[ndep]]][y___]->{{y},{x}}];

              Jlist = {};
              Jterm = Apply[Plus,Jcase[[1,2]] ];
              AppendTo[Jlist, Thread[ DeriList[Jcase[[1,1]],Jcase[[1,2]] ] ]];
              Jlist = Flatten[Jlist,1];
              Jlist = Delete[Jlist,0];

(* --- Equation (5.48) of Olver --- *)
            Phisum = delta[[i]]*D[ charac[[i]] ,
                         D[ dependVar[[ndep]][indepVar], Jlist ]];
            Phij = (-1)^Jterm*D[ Phisum, Jlist];
           AppendTo[Prlist,Phij],
           ],
```

```
         {i,1,Length[denslist]}];
      erg = erg + Apply[Plus,Prlist],
      {n,1,order}],
      {ndep,1,lendep}];
      erg = erg +
             Sum[delta[[ndep]]*D[charac[[ndep]],dependVar[[1]][indepVar]],
             {ndep,1,Length[charac]}];
      erg = Expand[erg]
      ];


(* --- Frechet derivative --- *)
(* --- Formulas are identical with (5.40) of             --- *)
(* --- P. Olver (1986) p. 318                            --- *)
(* ---
      order : determines the highest order of derivatives in the
              characteristics
                                                           --- *)

Frechet[delta_List, indVar_List, dependVar_List, characteristics_List,
              order_] := Block[
{lind,indepVar,lenin,lendep,dlist,d1list,Prlist,denslist,
 Jcase,Jlist,erg,Phij,Phisum,charac},

      DeriList[x_,y_]:={x,y};

(* --- Length of lists for independent and dependent variables --- *)
      lind = indVar;
      indepVar = Delete[lind,0];
      lenin = Length[lind];
      lendep = Length[dependVar];
      erg = 0;
      charac = characteristics;

(* --- Calculation of the Frechet derivative;
                         sum over dependent variables --- *)
      Do[
      dlist = {dependVar[[ndep]][indepVar]};
      d1list = charac;

(* --- Loop over orders --- *)
      Do[
      Prlist = {};
(* --- Determine all derivatives of the dependent variables up   --- *)
(* --- to order n and cancel all multiple terms                  --- *)
            dlist = Outer[D,dlist,{indepVar}];
            dlist = Union[Flatten[dlist]];

(* --- Calculate the derivatives with respect to the gradients --- *)
      denslist = Flatten[Outer[D,d1list,dlist]];
         Do[If[ Not[ denslist[[i]] === 0],
           Jcase = Cases[dlist,
                         Derivative[x___][dependVar[[ndep]]][y___]->{{y},{x}}];

              Jlist = {};
```

```
              AppendTo[Jlist, Thread[ DeriList[Jcase[[1,1]],Jcase[[1,2]] ] ]];
              Jlist = Flatten[Jlist,1];
              Jlist = Delete[Jlist,0];

(* --- Equation (5.40) of Olver --- *)
             Phij = D[ delta[[i]], Jlist];
             Phisum = Phij*D[ charac[[i]] ,
                          D[ dependVar[[ndep]][indepVar], Jlist ]];
             AppendTo[Prlist,Phisum]
            ],
          {i,1,Length[denslist]}];
      erg = erg + Apply[Plus,Prlist],
      {n,1,order}],
      {ndep,1,lendep}];
      erg = erg +
            Sum[delta[[ndep]]*D[charac[[ndep]],dependVar[[1]][indepVar]],
            {ndep,1,Length[charac]}];
      erg = Expand[erg]
      ];



(* --- Solve equations to find the substitutions --- *)

Substit[expr_List, subst_List, independent_List, order_]:=Block[
      {dlist,levels={},level,cases,result,var,derivlist,dindependent},

      result = Flatten[Solve[expr == 0, subst]];

(* --- Determine the exclusion variable --- *)
      Do[
      level = {result[[i,1]]};
      cases = Cases[level,Derivative[x___][u_][y___]->{u,{y},{x}}];
      AppendTo[levels,cases],
      {i,1,Length[result]}];

      var = {};
      Do[
         Do[
         If[levels[[i,1,3,j]] > 0, AppendTo[var,levels[[i,1,2,j]] ] ],
         {j,1,Length[levels[[1,1,2]] ]}],
      {i,1,Length[levels]}];
      var = Union[var];
      If[Length[var] === 1, exclude = var[[1]],
                            Return["   ### ERROR in Substit: no unique
                            exclusion variable"]
                            ];

(* --- Calculate the higher substitution rules --- *)
      If[Length[independent] === 1,
          dindependent = {exclude},
          dindependent = DeleteCases[independent,exclude]
      ];
      dlist = result;
      derivlist = {exclude};
```

```
      AppendTo[derivlist,result];

(* --- All derivatives up to order ORDER --- *)
            Do[dlist = Outer[D,dlist,dindependent];
               AppendTo[derivlist,dlist],
            {i,1,order}];
      derivlist = Flatten[derivlist];
      derivlist = Union[derivlist]
      ]


(* --- Search for the coefficients --- *)

Searchcof[expr_, dependent_List, order_]:=Block[{level,lenlev,cf1,ExprH},
      ExprH = Expand[expr];
(* --- Test if single term or sum --- *)
      If[FreeQ[ExprH,Plus],level = {ExprH},
         level = Level[ExprH,1]
      ];
      lenlev = Length[level];
(* --- Put the derivatives of the dependent variables in a list --- *)
      terms = {};
      Do[AppendTo[terms,Searchf2[ level[[i]], dependent, order ] ], {i,1,lenlev}];
      terms = Flatten[terms];
      terms = Reverse[Union[terms]];
      terms = Sort[terms,PowerT[#1] > PowerT[#2]&];
(* --- Determine the coefficients --- *)
      listcof = {};
      Do[cf1 = Coefficient[ ExprH,terms[[i]] ];
        AppendTo[listcof,cf1];
        ExprH = Expand[ExprH - cf1*terms[[i]] ],
      {i,1,Length[terms]}];
      If[Not[ExprH === 0],AppendTo[listcof,{ExprH}]];
      listcof = DeleteCases[listcof,0];
      listcof = Flatten[listcof]
      ]

(* --- Match the derivatives --- *)
Searchf1[expr_, dependent_List, order_] :=Block[{case,var,arg,der},
    fs1[x_,y_]:={x,y};
    case = Cases[{expr}, (mult_.)*Derivative[x___][u_][y___]^(n_.) ->
    {u, mult, n, {y}, {x}}];
    If[case != {} && Not[Intersection[dependent,{case[[1,1]]}] == {}] &&
    Apply[Plus, case[[1,5]] ] > order,
    var = Delete[Thread[fs1[case[[1,4]],case[[1,5]] ] ] ,0];
    arg = case[[1,1]][Delete[case[[1,4]],0]];
    der = D[arg,var]^case[[1,3]],
    {}]
    ]

(* --- Extract the derivatives in u,v,... ---*)
Searchf2[expr_, dependent_List, order_]:=Block[{level,lenlev,ExprHi,ExprH,result},
      level = Level[expr,1];
      lenlev = Length[level];
      ExprHi = expr;
```

```
      result = 1;
      Do[ExprH = Searchf1[ExprHi,dependent,order];
         If[Not[ExprH === {}],
         ExprHi = Expand[ExprHi/ExprH];
         result = result*ExprH
         ],
      {i,1,lenlev}];
      result
      ]


(* --- Power of derivatives --- *)
PowerT[expr_] := Block[{ExprH,caseP,levelP},
   ExprH = expr;
   caseP ={};
     If[FreeQ[ExprH, Times], levelP = {ExprH},
        levelP = Level[ExprH, 1]];
        Do[AppendTo[caseP,
        Cases[{levelP[[i]]}, (mult_.)*Derivative[x___][u_][y___]^(exp_.) ->
          {exp}]], {i, 1, Length[levelP]}];
      caseP = Flatten[caseP];
      Apply[Plus, caseP]]

(* --- substitution rule --- *)
SubstRule[x_,y_]:=x->y


(* --- Simplification of determining equations --- *)

Simplifyeqns[LExpr_List,characteristics_List,order_,warnings_]:=Block[
      {lExpr,lenExpr,L1,L2},
(* --- Unify the list LExpr --- *)
      lExpr = Union[LExpr];
(* --- Searching monomials in the list lExpr and put them in Result1 --- *)
(* --- Elements containing  Plus are collected in list Result2      --- *)

      lenExpr = Length[lExpr];
      Result1 = {};
      Result2 = {};
      Do[If[FreeQ[lExpr[[i]], Plus], AppendTo[Result1,lExpr[[i]] ],
                                     AppendTo[Result2,lExpr[[i]] ]],
      {i,1,lenExpr}];

(* --- Extract factors from the elements in list Result1 and create --- *)
(* --- substitution rules in auxiliary list L1                      --- *)
      L1 = {};
      Result1 = CancelPreFactor[Result1,characteristics,warnings];

      L1 = Table[0,{i,1,Length[Result1]}];
      L1 = Thread[SubstRule[Result1,L1]];
(* --- Apply the substitutions on list Result2 --- *)
      Result2 = Result2 /. L1;
      AppendTo[Result2,Result1];
      Result2 = DeleteCases[Result2,0];
      Result2 =Flatten[Result2];
```

```
      Result1 = {};
      Do[If[FreeQ[Result2[[i]],Plus] && Head[Result2[[i]] ] == Times,
            AppendTo[Result1,Result2[[i]]]
          ],
      {i,1,Length[Result2]}];

      L1 = {};
      Result1 = CancelPreFactor[Result1,characteristics,warnings];
      L1 = Table[0,{i,1,Length[Result1]}];
      L1 = Thread[SubstRule[Result1,L1]];
(* --- Apply the substitutions on list Result2 --- *)
      Result2 = Result2 /. L1;
      AppendTo[Result2,Result1];
      Result2 = DeleteCases[Result2,0];
      Result2 =Flatten[Result2];

      Result2 = Union[Result2];
      Result2 = Sort[Result2,FreeQ[#1,Plus]&];

(* --- Elimination of derivatives --- *)
      Do[ Elimin[i,characteristics,warnings],
      {i,1,order}];
      Result2

      ]

(* --- Extract factors from the elements in list expr  --- *)
CancelPreFactor[expr_List,characteristics_List,warnings_]:=Block[
                   {Result1,Lhelp,levels,levels1,fac1,lh,
                    characteristicsIn},
      Result1 = expr;
      Lhelp = {};
      characteristicsIn = characteristics;
      Do[If[FreeQ[Result1[[i]], Times],
            AppendTo[Lhelp,Result1[[i]] ],

            levels = Level[Result1[[i]],1];
            levels1 = {};

          Do[
             If[Head[levels[[j]] ] == Integer ||
                     Head[levels[[j]] ]== Real ||
                     Head[levels[[j]]] == Rational ||
                     Head[levels[[j]]] == Symbol,
                     AppendTo[levels1,levels[[j]] ] ],
          {j,1,Length[levels]}];

          fac1 = Apply[Times,levels1];

          If[fac1 === 0, lh = Result1[[i]],
                         lh = Result1[[i]]/fac1
          ];
(* --- give WARNINGS --- *)
              If[warnings && Not[NumberQ[fac1]],
```

```
            Print[" -- WARNING ! -- "];
            Print[" We eliminated the factor         ",fac1];
            Print[" which was the coefficient of     ",lh ];
            Print["  "]
          ];
        AppendTo[Lhelp,lh]
      ],
    {i,1,Length[Result1]}];
    Lhelp]



(* --- Replacement of higher derivatives if a monomial exists ---*)
Elimin[der_,characteristics_List,warnings_]:=Block[
      {LE1,LE2,LE3,sum1,sum,icount,level,case1,var1,var2,arg,lenRes2old,
      Result1},
      lenRes2old = 0;
      While[Length[Result2] != lenRes2old,
      lenRes2old = Length[Result2];
      Result1 = {};
      LE2 = {};
      LE3 = {};
      Do[If[FreeQ[Result2[[i]],Plus], AppendTo[Result1,Result2[[i]] ] ],
      {i,1,Length[Result2]}];

      Result1 = CancelPreFactor[Result1,characteristics,warnings];

      LE1 = Cases[Result1,Derivative[x___][u_[k_]][y___]->{u,k,{y},{x}}];
      Do[   nderi = der;
            sum1 = LE1[[i,4]];
            icount = 0;
            Do[If[sum1[[k]] > 0, icount = icount + 1],{k,1,Length[sum1]}];

        If[icount == 1,

            Do[If[FreeQ[Result2[[k]],Plus], level = {Result2[[k]]},
                  level = Level[Result2[[k]],1]];
                  case1 = Cases[level,
                          mult_.*Derivative[x___][u_[l_]][y___]->
                          {u,l,mult,{y},{x}}];
                Do[If[LTest[ LE1[[i]],case1[[m]] ],
                     var1 = Delete[case1[[m,5]],0];
                     var2 = Delete[case1[[m,4]],0];
                     arg = case1[[m,1]][ case1[[m,2]] ];
                     AppendTo[LE3,Result1[[i]]];
                     AppendTo[LE2,Derivative[var1][arg][var2]->0]],
                {m,1,Length[case1]}],
            {k,1,Length[Result2]}],
          Result2 = Result2 /. Result1[[i]]->0;
          AppendTo[Result2,Result1[[i]]]
        ],
      {i,1,Length[LE1]}];
      LE2 = Union[LE2];
      Result2 = Result2 /. LE2;
      Result2 = AppendTo[Result2,LE3];
```

```
      Result2 = Flatten[Result2];
      Result2 = DeleteCases[Result2,0];
      Result1 = {};
      Do[If[FreeQ[Result2[[i]],Plus],
            AppendTo[Result1,Result2[[i]] ]
          ],
      {i,1,Length[Result2]}];
      Result1 = CancelPreFactor[Result1,characteristics,warnings];
      LE2 = Table[0,{i,1,Length[Result1]}];
      LE2 = Thread[SubstRule[Result1,LE2]];
      Result2 = Result2 /. LE2;
      AppendTo[Result2,Result1];
      Result2 = Flatten[Result2];
      Result2 = DeleteCases[Result2,0];
      Result2 = Union[Result2];
      Result2 = Sort[Result2,FreeQ[#1,Plus]&]
      ]
      ]

(* --- Test derivatives if identical  --- *)
LTest[x_List, y_List] := Block[{t1},
  t1 = False;
(* --- Test same variables --- *)
     If[x[[1]] === y[[1]] && x[[2]] === y[[2]],
         Do[If[x[[4,i]] == nderi && x[[4,i]] <= y[[5,i]],
         t1 = True],
         {i, 1, Length[x[[4]]]}]
      ];
     t1]

(* --- Calculate the determining equations if information   --- *)
(* --- on characteristics are given                         --- *)


Substitution[expr_List,characteristics_List,charexpression_List]:=Block[
      {ListS,level,simpli,case,var,ExprH,exprIn,
       characteristicsIn,charexpressionIn,Listu,DependVar},

      DeriList[x_,y_]:={x,y};

      characteristicsIn = characteristics;
      charexpressionIn = charexpression;
(* --- Declare substitutions for dependent variables --- *)
      DependVar = {Delete[characteristicsIn[[1]],0]};

      Listu = {};
      Do[AppendTo[Listu,Unique["VAR"]],{i,1,Length[DependVar]}];
      ListS = Thread[SubstRule[DependVar,Listu]];
      charexpressionIn = charexpressionIn /. ListS;
      characteristicsIn = characteristicsIn /. ListS;
      Lout = {};
      Do[
      exprIn = expr[[iexp]];
      ExprH = Expand[exprIn] /. ListS;
```

```
(* --- Determine the cases of ExprH --- *)
      If[FreeQ[ExprH,Plus],level = {ExprH},
                           level = Level[ExprH,1]];
(* --- Find the derivatives in ExprH --- *)
      simpli = 0;

      Do[
         case = {};
         If[FreeQ[level[[i]],Derivative], simpli = simpli + level[[i]],
            case = Cases[{level[[i]]},mlt_.*Derivative[x___][u_][y___]->
                         {u,mlt,{y},{x}}];
           If[Length[case] > 0,
            Do[
            If[case[[1,1]] === Head[ characteristicsIn[[j]] ],
               dep = charexpressionIn[[j]] ],
            {j,1,Length[characteristicsIn]}];
            var = Thread[DeriList[ case[[1,3]], case[[1,4]] ] ];
            var = Delete[var,0];
(* --- Determine the result from ExprH --- *)
            simpli = simpli + case[[1,2]]*D[dep,var]
           ]
         ],
      {i,1,Length[level]}];
(* --- Do substitutions --- *)
      simpli = simpli /. Thread[SubstRule[characteristicsIn,charexpressionIn]];
      simpli = simpli /. Thread[SubstRule[Listu,DependVar]];
      simpli = Expand[simpli];
      AppendTo[Lout,simpli],
      {iexp,1,Length[expr]}];
      Lout = Expand[Lout]
      ]


AnsatzPolynom[variables_List,independVar_List,orderDepend_,orderDerivative_]:=
      Block[{listconst,erg,listpoly,lVariables,lResult,poly,ph,
      listderi},
      listconst = {};
      variab = variables;
      erg = 0;
         lVariables = {};
         Do[
            AppendTo[lVariables,variab[[iv]][Delete[independVar,0]]],
         {iv,1,Length[variables]}];
      Do[
      listpoly = {};
      listderi = {};
      Do[
         poly = 0;
         lResult = Table[1,{i,1,Length[variables]}];
         Do[
            lResult = Union[Flatten[Outer[Times,lResult,lVariables]]];
            lconstants = {};
            Do[AppendTo[lconstants,Unique["k"]],{j,1,Length[lResult]}];
            AppendTo[listconst,lconstants];
            poly = poly + lconstants.lResult,
```

```
      {i,1,orderDepend}];
     ph = Unique["k"];
     AppendTo[listconst,ph];
     poly = poly + ph;
     AppendTo[listpoly,poly],
    {k,1,Length[variables]^2}];
    listpoly = Partition[listpoly,Length[variables]];
    If[ld ===0, listderi = Table[1,{ik,1,Length[variables]}],
                Do[
                AppendTo[listderi,D[lVariables[[lv]],independVar[[1]]]^ld],
                {lv,1,Length[variables]}]
    ];
    erg = erg + listpoly.listderi,
    {ld,1,orderDerivative}];
    {erg,Flatten[listconst]}
 ]


(* --- Determine the coefficients of the Ansatz --- *)

Ansatzcof[coef_List,dependent_List,independent_List]:=Block[
    {coefficientList = {}, variables, coefIn, var, lenvar, expolist,
    exphl, termlist, term, coeff},

    variables = dependent;
    coefIn = coef;
    var = {};
    Do[
       AppendTo[var,variables[[i]][Delete[independent,0]]],
    {i,1,Length[dependent]}];
    lenvar = Length[var];
    Do[
       AppendTo[var,D[var[[i]],Delete[independent,0]]],
    {i,1,lenvar}];
    variables = var;
    var = {};
(* --- create variables for replavement --- *)
    Do[
       AppendTo[var,Unique["CKGB"]],
    {i,1,Length[variables]}];
(* --- replace the originals by var --- *)
    coefIn = coefIn /. Thread[SubstRule[variables,var]];
    Do[
(* --- determine the exponents of variables var --- *)
    expolist = Exponent[coefIn[[i]],var,List];
(* --- reorganize the lists --- *)
          exphl = {};
          Do[
             exph = {};
             Do[
                AppendTo[exph,expolist[[j,k]] ]
             ,{j,1,Length[var]}];
             AppendTo[exphl,exph]
          ,{k,1,Length[expolist[[1]]]}];
(* --- unify the exponents --- *)
```

```
      expolist = Union[exphl];
(* --- sort the exponents in decreasing order --- *)
      expolist = Sort[expolist,(Apply[Plus,#1] > Apply[Plus,#2])&];
      Do[
      termlist = {};
      Do[
          AppendTo[termlist,var[[it]]^expolist[[j,it]] ]
      ,{it,1,Length[var]}];
      term = Apply[Times,termlist];
      coeff = Coefficient[coefIn[[i]],term];
      AppendTo[coefficientList,coeff];
      coefIn[[i]] = Expand[coefIn[[i]]-coeff*term];
      ,{j,1,Length[expolist]}];
      AppendTo[ coefficientList,coefIn[[i]] ]
      ,{i,1,Length[coefIn]}];
      coefficientList = Union[coefficientList];
      coefficientList = DeleteCases[coefficientList,0]
      ];


(* --- Determine monomial terms --- *)

Monomials[coefficientList_List]:=Block[{monomials={}},
      Do[If[FreeQ[coefficientList[[i]],Plus],
            AppendTo[monomials,coefficientList[[i]]] ],
      {i,1,Length[coefficientList]}];
      monomials
      ]


(* --- Determine the constants of the Ansatz --- *)

DetCoeff[coefficient_List,ansatz_List,para_List]:=Block[
      {ansatzIn,monomials={},var,nul,coefficientList,
      var1,equationlist = {},result,rs1help,v1k,parahelp,paratest},

      coefficientList = coefficient;
      ansatzIn = ansatz;
(* --- extract terms free of Plus --- *)
      monomials = Monomials[coefficientList];
(* --- determine variables in the single terms --- *)
      While[Length[monomials] > 0,
      var = Complement[Variables[monomials],para];
      nul = Table[0,{i,1,Length[var]}];
      var = Thread[SubstRule[var,nul]];
(* --- replace single terms by zero --- *)
      coefficientList = coefficientList /. var;
      ansatzIn = ansatzIn /. var;
      coefficientList = DeleteCases[coefficientList,0];
      monomials = Monomials[coefficientList]
      ];

      var1 = Variables[coefficientList];

(* --- eliminate numeric factors in single term equations --- *)
```

```
         singleTerm = Factor[coefficientList];
         singleFactor = {};
         Do[
             If[FreeQ[singleTerm[[i]],Times],
                AppendTo[singleFactor,singleTerm[[i]] ],
                level = Level[singleTerm[[i]],1];
                   Do[
                      If[NumberQ[level[[j]]] || ParameterQ[level[[j]]],
                      singleTerm[[i]] = Expand[singleTerm[[i]]/level[[j]] ]
                      ],
                   {j,1,Length[level]}];
               AppendTo[singleFactor,singleTerm[[i]] ]
               ],
         {i,1,Length[singleTerm]}];

         coefficientList = Flatten[Union[singleFactor]];

(* --- use the reduced equations to determine the remaining relations --- *)
       Do[AppendTo[equationlist,coefficientList[[i]]==0],
       {i,1,Length[coefficientList]}];
       Print["     calling Solve with ",Length[equationlist]," equations and
       ",Length[var1]," variables"];
       (*result = Solve[equationlist,var1];*)
                v1k=Reduce[equationlist,var1];
                v1k = ToRules[v1k];
                result = {v1k};
       result = Union[result];
       resansatz = {};
       Do[
          parahelp = para /. result[[i]];
          paratest = Apply[Plus,parahelp];
          If[Not[paratest === 0],
          rs1help = ansatzIn /. result[[i]];
          AppendTo[resansatz,{rs1help,parahelp} ]
          ],
       {i,1,Length[result]}];
       resansatz
       ];


(* --- check input as an element in the list ParameterS --- *)
ParameterQ[input_]:=Block[{},
       If[Length[Complement[parameteR,{input}]] < Length[parameteR],
          True,
          False
       ]
       ]






(* --- Defaultvalues for options --- *)
```

```
Options[Baecklund] = {Warnings->True,
                      DerivativeOrder->1,
                      CharExpression->{},
                      AnsatzPoly->{0,0},
                      VariationalCheck->False
                      }


(* --- Baecklund --- *)

Baecklund[expr_List, subst_List, independent_List, dependent_List,
          order_,para_List,options___]:=Block[
      {characteristicsList = {},prolList = {},cofList = {},
      ansatzPoly = {},ansatz,
      frechetList,proList, proposition, propSum, density, integral,
      substitu,exclude,warnings,derivativeOrder,infoGiven,
      charexpressionList,parameterIn,relations,rules,ruleS},

      parameterIn = para;
(* --- Setup the options --- *)
      {warnings,derivativeOrder,charexpressionList,ansatzPoly,
       variationalCheck} =
      {Warnings,DerivativeOrder,CharExpression,AnsatzPoly,
       VariationalCheck}
                                                  /. {options}
                                                  /. Options[Baecklund];
(* --- Check options --- *)
      If[charexpressionList === {}, infoGiven = False,
                                    infoGiven = True];

(* --- Determine the substitutions --- *)
      t1 = Timing[
      substitu = Substit[expr, subst, independent, order];

      If[Length[independent] === 1, exclude = {},
                                    exclude = substitu[[1]] ];
      substitu = Delete[substitu,1];
      Print["   "];
      Do[
      Print["  Substitution No. ",i];
      Print["   "];
      Print["  ",substitu[[i]] ];
      Print["   "],
      {i,1,Length[substitu]}];
Print["  -----------------------------------------------------------------"];
      If[infoGiven,
Print["  -----------------------------------------------------------------"];
Print["  Characteristics :"];
      Do[
      Print["   "];
      Print["  Q",i," = ",charexpressionList[[i]]],
      {i,1,Length[charexpressionList]}];
Print["  -----------------------------------------------------------------"]
       ]
      ];
```

```
     Print["   Determinig the Substitutions                  ", t1[[1]] ];
(* --- Create characteristics --- *)
     t1 = Timing[
     AppendTo[characteristicsList,
              Characteristics[independent, dependent, order, exclude]];
     characteristicsList = Flatten[characteristicsList]
     ];
     Print["   Creating the Characteristics                  ", t1[[1]] ];

(* --- Determination of the highest derivatives --- *)
     MaxDeri = {};
     Do[
       Do[
          AppendTo[MaxDeri,MaxD[expr[[i]],dependent[[j]]]],
       {i,1,Length[expr]}],
     {j,1,Length[expr]}];
     MaxDeri = Max[MaxDeri];
     HighestDerivatives = MaxDeri;

(* --- Calculate generalized prolongations --- *)
     t1 = Timing[
     Do[
     AppendTo[prolList,
     Prolongation[expr[[i]], independent, dependent, characteristicsList,
     HighestDerivatives]],
     {i,1,Length[expr]}]
     ];
     Print["   Calculate generalized prolongations           ", t1[[1]] ];

(* --- Separate numerator and denominator --- *)
     t1 = Timing[
     prolList = Together[prolList];
     prolList = Cancel[prolList];
     Denum = Denominator[prolList];
     prolList = Expand[Numerator[prolList]]
     ];
     Print["   Separate numerator and denominator            ", t1[[1]] ];

(* --- Make the substitutions of higher order derivatives --- *)
     t1 = Timing[
     prolList = prolList //. Reverse[substitu];
     prolList = Expand[prolList]
     ];
     Print["   Substitute the higher order derivatives       ", t1[[1]] ];


(* --- Determine the coefficients of the prolongation --- *)
     t1 = Timing[
     Do[
     AppendTo[cofList,Searchcof[prolList[[i]], dependent, order] ],
     {i,1,Length[prolList]}];
     cofList = Flatten[cofList]
     ];
     Print["   Determine the coefficients of the prolongation ", t1[[1]] ];
```

```
(* --- Simplification of the determining equations --- *)
      Print["   Entering simplification "];
      t1 = Timing[
      Print["   The number of equations before simplification "];
      Print["   *** N_before: ",Length[cofList] ];
      Print["    "];
      cofList = Simplifyeqns[cofList,characteristicsList,order,warnings];
      Print["    "];
      Print["   The number of equations after simplification "];
      Print["   *** N_after: ",Length[cofList] ];
      Print["    "]
      ];
      Print["   Simplification finished                       ", t1[[1]] ];

(* --- Check the given characteristics --- *)
      If[!infoGiven,cofList,
      t1 = Timing[
           cofList = Substitution[cofList,characteristicsList,
                                  charexpressionList];
           cofList = DeleteCases[cofList,0]
      ];
      Print["   Inserting the given information              ", t1[[1]] ]
      ];
(* --- Make an Ansatz for ODE's of second order --- *)
      If[ansatzPoly[[1]] > 0 && ansatzPoly[[2]] >= 0 &&
         Length[independent] === 1,
         Print["   Make an Ansatz for ODE's of second order"];
         t1 = Timing[
         ansatz = AnsatzPolynom[dependent,independent,ansatzPoly[[1]],
                                    ansatzPoly[[2]]]
                     ];
         a1h = ansatz;
         Print["     create an Ansatz :                          ",t1[[1]] ];
         t1 = Timing[
         cofList = Substitution[cofList,characteristicsList,
                                   ansatz[[1]] ]
                    ];
         Print["     substitute the Ansatz :                     ",t1[[1]] ];
         t1 = Timing[
         cofListAnsatz = Ansatzcof[cofList,dependent,independent]
                    ];
         Print["     resulting equations from the Ansatz :       ",t1[[1]] ];

         t1 = Timing[
         relations = DetCoeff[cofListAnsatz,ansatz[[1]],parameterIn];
                    ];
         Print["     determine Coefficients of the Ansatz :      ",t1[[1]] ];
         cofList = relations
      ];
(* --- Check the given characteristics to be variational --- *)
      If[variationalCheck ||
         (ansatzPoly[[1]] > 0 && ansatzPoly[[2]] >= 0 &&
         Length[independent] === 1),
(* --- convert the results from the ansatz to different lists --- *)
              If[ ansatzPoly[[1]] > 0 && ansatzPoly[[2]] >= 0 &&
```

```
                Length[independent] === 1,
                charexpressionList = {};
                rules = {};
                depend = {};

          Do[AppendTo[depend,DependVar[[i]][Delete[IndepVar,0] ] ],
          {i,1,Length[DependVar]}];
                AppendTo[depend,D[depend,IndepVar[[1]]] ];
                depend = Flatten[depend];

                Do[
                   AppendTo[rules,cofList[[i,2]] ];
                   AppendTo[charexpressionList,cofList[[i,1]] ],
                {i,1,Length[cofList]}]
              ];

       Print["   "];
       Print["   Check variational symmetry"];
       t1 = Timing[
    variational = {};
    varIn = {};
    Do[
    constants = Complement[
                 Variables[charexpressionList[[ichar]]],depend,parameteR];
    crules = ConstantCases[constants];
(* --- create rules --- *)
    ruleS = Thread[SubstRule[parameteR,rules[[ichar]]] ];
      Do[
      charexpressionL = charexpressionList[[ichar]] /. crules[[irule]];
      charexpressionL = charexpressionL /. ruleS;
      integral = 0;
      AppendTo[varIn, expr /. ruleS];
      AppendTo[variational,{charexpressionL,rules[[ichar]],integral}],
      {irule,1,Length[crules]}],
    {ichar,1,Length[cofList]}];
  Do[
(* --- variational check --- *)
    proList = {};
    Do[
    AppendTo[proList,
    Prolongation[varIn[[ichar,i]], independent, dependent,
    variational[[ichar,1]],HighestDerivatives]],
    {i,1,Length[varIn[[ichar]]]}];

    frechetList = {};
    Do[
    AppendTo[frechetList,
    adjointFrechet[varIn[[ichar]], independent, {dependent[[i]]},
    variational[[ichar,1]], HighestDerivatives]],
    {i,1,Length[varIn[[ichar]]]}];

(* --- determine the result from Proposition 5.39 p. 326 of Olver --- *)
    proposition =  Expand[frechetList + proList];
    proposition = Expand[proposition];
    propSum = Apply[Plus,proposition];
```

```
(* --- calculate the corresponding integral of motion --- *)
          If[propSum === 0,
(* --- Variational check satisfied  --- *)
          density = varIn[[ichar]].variational[[ichar,1]];
          integral = Integrate[density,Delete[independent,0]];
          variational[[ichar,3]] = integral,
(* --- Variational check failed --- *)
          variational[[ichar,3]] =  0
          ],
  {ichar,1,Length[variational]}]
         ];
         cofList = variational;
         Print["     Variationalcheck finished                    ", t1[[1]] ]
      ];
      cofList
      ]


(* --- Create substitution lists for variational check --- *)
      ConstantCases[constants_]:=Block[{rules,ruleshelp},
      rules = {};
      Do[
        ruleshelp = {};
        Do[
         If[i == j,
            AppendTo[ruleshelp,constants[[i]]->1],
            AppendTo[ruleshelp,constants[[i]]->0]
            ],
        {i,1,Length[constants]}];
        ruleshelp = Flatten[ruleshelp];
        AppendTo[rules,ruleshelp],
      {j,1,Length[constants]}];
      rules
      ]
```

```
(* --- Defaultvalues for the options in batchmode --- *)

Options[LieBaecklund] = {Warnings->True,
                      DerivativeOrder->1,
                      CharExpression->{},
                      AnsatzPoly->{0,0},
                      VariationalCheck->False
                      }


(* --- Batch mode command --- *)
```

```
LieBaecklund[name_String,options___]:=Block[{filetest,optin,optout,
      warnings,charexpressionList,ansatzPoly,variationalCheck,
      derivativeOrder,
      Warnings,CharExpression,AnsatzPoly,VariationalCheck,DerivativeOrder,
      ansatz1,ansatz2,logFilename,teXfilename,nameSystem,nameHelp},

      filetest = FileNames[name];
      If[Not[filetest === {}],

(* --- filename creation --- *)
      nameHelp = StringTake[name,-4];
      nameSystem = StringReplace[name,nameHelp->".deq"];
      teXfilename = StringReplace[name,nameHelp->".tex"];
      logFilename = StringReplace[name,nameHelp->".lba"];

(* --- Log calculations --- *)
      OpenLog[logFilename];

(* --- Loading data from batchfile --- *)
         Print[" "];
         Print[" Date :  ",Date[]];
         Print[" Loading data from  ",name];
         Print["  "];
         Get[name];

(* --- Setup the options --- *)
      optin = {warnings,derivativeOrder,charexpressionList,ansatzPoly,
       variationalCheck};
       optout = {Warnings,DerivativeOrder,CharExpression,AnsatzPoly,
       VariationalCheck};


      optin =
      {Warnings,DerivativeOrder,CharExpression,AnsatzPoly,
       VariationalCheck}
                                                 /. {options}
                                                 /. Options[LieBaecklund];

      ansatz1 = optin[[4,1]];
      ansatz2 = optin[[4,2]];
      optin = Delete[Thread[SubstRule[optout,optin]],0];


(* --- Output the equations --- *)
      Print[" "];
Print["  -----------------------------------------------------------------"];
      Do[
      Print["  Equation No. ",i];
      Print["   "];
      Print["  ",EqList[[i]]," = 0"];
      Print["   "],
      {i,1,Length[EqList]}];
Print["  -----------------------------------------------------------------"];
         Print[derivativeOrder];
         Print[optin];
```

```
         FinalResults = Baecklund[EqList,SubsList,IndepVar,DependVar,
               derivativeOrder,parameteR,optin],

            Print[" ### Error: File ",name," not found"];
            Return["Done"]];

(* --- Output the results --- *)
      If[FinalResults != {} && variationalCheck,
         PrintIntegrals[FinalResults];
         Save[nameSystem,FinalResults],
         If[ansatz1 > 0 && ansatz2 >= 0,
            PrintCharacteristic[FinalResults];
            Save[nameSystem,FinalResults],
            PrintEquations[FinalResults];
            Save[nameSystem,FinalResults]
           ]
      ];
      If[FinalResults === {} ,
Print["  -------------------------------------------------------------------"];
Print["                        Equations satisfied!"];
Print["  -------------------------------------------------------------------"]
      ];
      CloseLog[];
      Return["Done"]
      ]


(* --- Print integrals --- *)

PrintIntegrals[result_List]:=Block[{},
         Print["  "];
         Do[Print["  Integral No. : ",i];
Print["  -------------------------------------------------------------------"];
            Print["  ","I = ",result[[i]] ];
            Print[" "];
Print["  -------------------------------------------------------------------"],
         {i,1,Length[result]}]
         ]


(* --- Print equations --- *)

PrintEquations[result_List]:=Block[{},
         Print[" "];
         Do[
           Print[" "];
           Print["  Equation No. : ",i];
Print["  -------------------------------------------------------------------"];
            Print["  ",result[[i]]," = 0" ];
            Print[" "];
Print["  -------------------------------------------------------------------"],
         {i,1,Length[result]}];
         Print[" "];
Print["  **** The determining equations are stored in FinalResults.  ****"]
         ]
```

```
(* --- Print characteristics --- *)

PrintCharacteristic[result_List]:=Block[{},
        Print[" "];
        ik = 0;
        Do[
           If[Not[result[[i,3]]===0],
           ik = ik + 1;
           Print[" "];
           Print["  Characteristic No. : ",ik];
Print["  ------------------------------------------------------------------"];
           Print["  ",result[[i,1]] ];
           Print[" "];
           Print["  Parameter: ",parameteR," = ",result[[i,2]]];
           Print[" "];
Print["  ------------------------------------------------------------------"];
           Print["  I = ",result[[i,3]] ];
           Print[" "];
Print["  =================================================================="]
           ],
        {i,1,Length[result]}];
        Print[" "];
        Print["  **** The characteristics are stored in FinalResults. ****"];
        Print["  "]
        ]


(* --- Logging input and output --- *)

OpenLog[filename_String]:=(
      logfile = OpenWrite[filename, FormatType->OutputForm];
      If[logfile === $Failed, Return[] ];
      AppendTo[$Echo, logfile];
      AppendTo[$Output, logfile]
      )
CloseLog[]:=(
      $Echo = Complement[$Echo, {logfile}];
      $Output = Complement[$Output, {logfile}];
      Close[logfile];
      logfile = " "
      )

Print["    "]
Print["  LieBaecklund loaded. Call it by LieBaecklund[name,options]"]
Print["    "]

End[]
EndPackage[]
```