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Indefinite and Definite Integration

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1 Indefinite Integration

1.1 Differential Fields

Internal Representation. The first step in *Mathematica*'s integration algorithm is to convert the integrand into an internal form where recognition of algebraic relationships and lack of such relationships is comparatively easy. For example, the input

$$x\, e^x \cos x + \sqrt{x+1} + \sqrt[3]{\sinh x^2} + \log x + \sin^{-1} x$$

is converted into the following form:

$$v_4\, p_5 \left(\frac{c_6}{2} + \frac{{c_6}^{-1}}{2}\right) + a_7 + a_9 + v_{10} + v_{12}$$

where the definitions of the internal variables are given by

$$\begin{array}{ll}
\operatorname{def}(v_1) = 1 & \operatorname{def}(v_2) = e \\
\operatorname{def}(v_3) = \pi & \operatorname{def}(v_4) = x \\
\operatorname{def}(p_5) = {v_2}^{v_4} & \operatorname{def}(c_6) = {v_2}^{i\, v_4} \\
\operatorname{def}(a_7) = \sqrt{v_4 + 1} & \operatorname{def}(h_8) = {v_2}^{{v_4}^2} \\
\operatorname{def}(a_9) = \sqrt[3]{\dfrac{h_8}{2} - \dfrac{{h_8}^{-1}}{2}} & \operatorname{def}(v_{10}) = \log v_4 \\
\operatorname{def}(a_{11}) = \sqrt{1 - {v_4}^2} & \operatorname{def}(v_{12}) = -i \log(i\, v_4 + a_{11})
\end{array}$$

Some facts—such as the transcendental independence of v_4, p_5, c_6, v_{10}, and v_{12} and that a_7, a_9 are algebraics—can be read directly off internal form. Other facts—such as $v_4' = 1, p_5' = p_5, v_{10}' = v_4^{-1}$, or $a_7^2 = v_4 + 1$—are recorded on the side.

There are five kinds of indexed variables: a_i, c_i, h_i, p_i, and v_i. Examples of how these variables are used are given below:

1) a_i, *algebraics*: $\sqrt{x+1}, \sqrt[3]{y}, \sqrt{\sin z}$

2) c_i, *circulars*: $\cos x, \sin u^2, \tan(e^v)$

3) h_i, *hyperbolics*: $\cosh x, \sinh u^2, \tanh(e^v)$

4) p_i, *powers*: $e^x, y^y, a^{\log x}$

5) v_i, *other variables*: $x, y, z, \log x, \sin^{-1} z, e, \pi$

There is an input routine and an output routine that convert into and out of this representation. Otherwise, the operations of *Mathematica*'s integrator are performed exclusively on internal forms. The rest of this paper will concern itself with how this internal representation is built up and then processed to produce answers.

Theta Variables. Throughout this paper, we use variables like $\theta_1, \ldots, \theta_n$ to stand for internal variables a_i, c_i, h_i, p_i, and v_i. We assume as a convention that $\operatorname{def}(\theta_1) = x$.

Differential Fields. A *differential field* F is a field F where, in addition to the usual operations of $+$, $-$, $\cdot$, and $/$, there is a derivation operator $'$, which maps F into itself and enjoys the properties

1) $(f+g)' = f' + g'$

2) $(f \cdot g)' = f' \cdot g + f \cdot g'$

We shall generally assume $x \in F$ and $x' = 1$, so that the $'$ operator can be thought of as $\partial/\partial x$. A *constant field* is a differential field K such that $f' = 0$ for all $f \in K$.

Elementary Fields. Let $F = K(\theta_1, \ldots, \theta_n)$ be a differential field where K is a constant field. Let $F_i = K(\theta_1, \ldots, \theta_i)$. Then we will say F is an *elementary field* provided that $\theta_1' = 1$ and for $2 \le i \le n$ one of the following three conditions holds:

1) θ_i is algebraic over F_{i-1}

2) $\theta_i' = \eta'/\eta$ for some $\eta \in F_{i-1}$ $(\theta_i = \log \eta)$

3) $\theta_i' = \eta'\,\theta_i$ for some $\eta \in F_{i-1}$ $(\theta_i = e^{\eta})$

We shall generally be working with elementary fields. A *purely transcendental field* is an elementary field in which θ_i is transcendental over F_{i-1} for $1 \le i \le n$.

Elementary Functions. The virtue of the concept of an elementary field is that any elementary function—the type of function that appears in a typical small table of integrals—can be made to lie in an elementary field. The integral of an elementary function, if elementary, will lie in the same elementary field, possibly extended by some algebraic constants and new logarithms.

Every function that is built up out of constants x, $+$, $-$, $\cdot$, $/$, $\exp()$, and $\log()$ is elementary. The set of elementary functions includes powers, radicals, trigonometric functions, and inverse trigonometric functions as we can see from the following table:

$$
\begin{array}{ll}
u^v = e^{v \log u} & \sqrt[n]{z} = z^{\frac{1}{n}} = e^{\frac{1}{n}\log z} \\
\sin(z) = -\frac{i}{2}\,e^{ix} + \frac{i}{2}\,e^{-ix} & \cos(z) = \frac{1}{2}\,e^{ix} + \frac{1}{2}\,e^{-ix} \\
\sinh(z) = \frac{1}{2}\,e^{x} - \frac{1}{2}\,e^{-x} & \cosh(z) = \frac{1}{2}\,e^{x} + \frac{1}{2}\,e^{-x} \\
\sin^{-1}(z) = -i\,\log(i\,z + \sqrt{1-z^2}) & \cos^{-1}(z) = \frac{\pi}{2} - \sin^{-1}(z) \\
\tan^{-1}(z) = -i\,\log\left(\frac{1+i\,z}{\sqrt{1+z^2}}\right) & \tan^{-1}(u,v) = -i\,\log\left(\frac{u+i\,v}{\sqrt{u^2+v^2}}\right) \\
\csc^{-1}(z) = \sin^{-1}(\frac{1}{z}) & \sec^{-1}(z) = \cos^{-1}(\frac{1}{z}) \\
\cot^{-1}(z) = \tan^{-1}(\frac{1}{z}) & \sinh^{-1}(z) = \log(z + \sqrt{1+z^2}) \\
\cosh^{-1}(z) = \log\left(z + (z+1)\sqrt{\frac{z-1}{z+1}}\right) & \tanh^{-1}(z) = \log\left(\frac{1+z}{\sqrt{1-z^2}}\right) \\
\operatorname{csch}^{-1}(z) = \sinh^{-1}(\frac{1}{z}) & \operatorname{sech}^{-1}(z) = \cosh^{-1}(\frac{1}{z}) \\
\coth^{-1}(z) = \tanh^{-1}(\frac{1}{z}) &
\end{array}
$$

The Input Algorithm. When presented with an integrand f, the integration algorithm first converts f into internal form. It processes the input by working inside out, adding new variables θ_i and new definitions $\mathrm{def}(\theta_i)$ to its table of definitions as necessary. However, the input algorithm, also, as it processes its input, makes conclusions about related subexpressions of the input so that, roughly speaking, as few θ_i variables are created as necessary.

To give an example, consider the input expression

$$e^x + e^{-x}$$

It is much better for the input algorithm to create the table of definitions

$$\mathrm{def}(\theta_1) = x \qquad \mathrm{def}(\theta_2) = e^{\theta_1}$$

and to represent the input as $\theta_2 + {\theta_2}^{-1}$ than to create the table of definitions

$$\begin{array}{ll} \mathrm{def}(\theta_1) = x & \mathrm{def}(\theta_2) = e^{\theta_1} \\ \mathrm{def}(\theta_3) = e^{-\theta_1} & \end{array}$$

and represent the input as $\theta_2 + \theta_3$. It should be obvious that θ_3 is algebraically related to θ_2: $\theta_3 = {\theta_2}^{-1}$.

Transcendental versus Algebraic. Given an extension field $F(\theta)$, we say that θ is algebraic over F, provided there is some nonzero polynomial $p(z) \in F[z]$ such that $p(\theta) = 0$. Otherwise, we say that θ is transcendental over F.

The integration algorithm works best when it is working inside a field $F = K(\theta_1, \ldots, \theta_n)$ where each θ_i is transcendental over $F_{i-1} = K(\theta_1, \ldots, \theta_{i-1})$. Some capability to treat algebraic θ_i is nevertheless available in *Mathematica*. The ability to treat algebraics in full generality is hopefully something that can be accomplished some time in the future.

It is important for the integration algorithm to distinguish whether each θ_i is algebraic or transcendental over F_{i-1}. For example, if

$$\mathrm{def}(\theta_1) = x \qquad \mathrm{def}(\theta_2) = \sqrt{\theta_1 + 1}$$

then it is critically important to avoid dividing by $\theta_2^2 - \theta_1 - 1$. The Risch Structure Theorem gives a way to make this kind of distinction, algorithmically.

Risch Structure Theorem. The Risch Structure Theorem can be stated as follows. Let η lie in the differential field $F = K(\theta_1, ..., \theta_n)$. Let E and L be the sets of indices of exponentials and logarithms among $\theta_2, ..., \theta_n$ respectively. Assume that $\theta_i = \log \eta_i$ or $\theta_i = e^{\eta_i}$ for $2 \le i \le n$. Then

1) e^η is algebraic over F if and only if there exists a constant $C \in K$ and some rational numbers ρ_i such that

$$\eta = C + \sum_{i \in E} \rho_i\, \eta_i + \sum_{i \in L} \rho_i\, \theta_i$$

2) $\log \eta$ is algebraic over F if and only if there exists a constant $C \in K$ and some rational numbers ρ_i such that

$$\eta = C \prod_{i \in E} {\theta_i}^{\rho_i} \prod_{i \in L} {\eta_i}^{\rho_i}$$

This equation implies the equation

$$\frac{\eta'}{\eta} = \sum_{i \in E} \rho_i\, \eta_i' + \sum_{i \in L} \rho_i\, \frac{\eta_i'}{\eta_i}$$

which is somewhat easier to use.

The Risch Structure Theorem gives us an algorithm for determining whether e^η or $\log \eta$ is transcendental over F, because it is possible to form the equations above and then solve for the C and ρ_i or prove they do not exist.

Normalized and Reduced Fields. As a further preparation for the Risch algorithm, we require that our differential fields be normalized and reduced. When Risch first proposed his algorithm to integrate elementary functions, there were extra complications to solving differential equations of the type

$$y' + f\,y = g$$

that entailed expensive calculations of degree bounds and solving linear systems of equations.

Davenport (1986) has found a way out of these difficulties by requiring differential fields to be normalized and reduced. Suppose that at any point as we are constructing our differential field we want to extend the field $F = K(\theta_1, ..., \theta_n)$ by the exponential e^η. If η depends on ϕ and lies in $K(\theta_1, ..., \phi)$, let the series expansion of η with respect to ϕ be

$$\eta = \ldots + \eta_{-1}\,\phi^{-1} + \eta_0 + \eta_1\,\phi + \ldots$$

We say that e^η is *reduced*, provided η_1 is not a rational number or ϕ is not a logarithm. If this is not the case, then $\eta_1\,\phi = \rho \log \zeta$ and the expression $\zeta^\rho\, e^{\eta - \eta_1\,\phi}$ should be input in place of e^η.

We say that e^η is *normalized* provided that η_0 is independent of x. If this is not the case, then the expression $e^{\eta_0}\, e^{\eta-\eta_0}$ should be input in place of e^η.

We say that the differential field F is normalized and reduced provided that all its exponentials are normalized and reduced.

1.2 The Risch Algorithm

Risch (1969) presented the first real breakthrough in the area of symbolic integration by describing an algorithm to compute integrals of elementary functions provided the answers are elementary functions.

Partial Fraction Decompositions. A critical concept required to describe the Risch algorithm is the notion of a partial fraction decomposition.

Suppose $f \in F(\theta)$. Then a *partial fraction decomposition* of f with respect to θ is a representation of f as

$$f(\theta) = f_{poly}(\theta) + \sum_{i,j} \frac{f_{ij}(\theta)}{q_i(\theta)^j}$$

where $f_{poly}(\theta), f_{ij}(\theta), q_i(\theta) \in F[\theta]$ are polynomials in θ, $\deg(f_{ij}(\theta), \theta) < \deg(q_i(\theta), \theta)$ and the $q_i(\theta)$ are relatively prime and square-free.

For technical reasons, there is the following additional complication if θ is of the form $\theta = e^\eta$. In this case, we allow $f_{poly}(\theta)$ to be a generalized polynomial of the form

$$f_{poly}(\theta) = \sum_{i=-m}^{n} f_i\,\theta^i$$

and add the restriction that $\theta \not| q_i(\theta)$. (There are even more complications when $\theta = c_i$ or $\theta = h_i$, but these complications won't be discussed.)

In any circumstances, we refer to

$$f_{poly}(\theta)$$

as the *polynomial part* of f and to

$$\sum_{i,j} \frac{f_{ij}(\theta)}{q_i(\theta)^{\,j}}$$

as the *fraction part* of f. We further break down the polynomial part of f by referring to

$$\sum_{i=-m}^{-1} f_i\,\theta^i, \qquad f_0, \qquad \sum_{i=1}^{n} f_i\,\theta^i$$

as the *negative half*, the *zero part*, and the *positive half* of f_{poly}.

Liouville Structure Theorem. Suppose $f \in F = K(\theta_1, \theta_2, \ldots, \theta_n)$. Then according to the Liouville Structure Theorem, if the integral of f is an elementary function, then we must have

$$\int f\,dx = y + \sum c_i \log h_i$$

for some $y \in F(x)$, *constants* $c_i \in \bar{K}$, and $h_i \in \bar{K}F(x)$. This theorem, proved by Risch, is a generalization of a result due to Liouville. We refer to y as the *rational part* of the integral, and we refer to $\sum c_i \log h_i$ as the *logarithmic part* of the integral. (The decomposition is not necessarily unique, as it may be possible to move some terms back and forth between the rational part and the logarithmic part.)

Risch's Theorem. Risch's Theorem can be stated as follows. Let $F = K(\theta_1, \theta_2, \ldots, \theta_n)$ be a differential field. Then

1) If $f \in F$, then there is an algorithm to compute

$$\int f\,dx = y + \sum c_i \log h_i$$

where $y \in F(x), c_i \in \bar{K}$, and $h_i \in \bar{K}F(x)$.

2) If $f, g \in F$, then there is an algorithm to solve

$$y' + f\,y = g$$

where $y \in F$.

Risch's Algorithm. The proof of Risch's Theorem and the algorithm for computing indefinite integrals of elementary functions proceeds by induction on n.

Base Step. As the base step of the induction proof, consider the case $n = 0$. Then $F = K$ is a constant field. Here we have

1) If $f \in F$, then

$$\int f\,dx = f\,x$$

and $f\,x \in F(x)$.

2) If $f, g \in F$, then

$$y' + f\,y = g$$

implies $y = g/f$.

The Induction Hypothesis. To carry out the rest of Risch's algorithm, we assume the following induction hypothesis. Let $F_i = K(\theta_1, \ldots, \theta_i)$. Then we assume that

1) If $f \in F_i$, then there is an algorithm to compute

$$\int f\,dx = y + \sum c_j \log h_j$$

where $y \in F_i(x), c_j \in \bar{K}$, and $h_j \in \bar{K}F_i(x)$.

2) If $f, g \in F_i$, then there is an algorithm to solve

$$y' + f\,y = g$$

where $y \in F_i$.

The Induction Step. The induction step will be to show that if the induction hypothesis holds for the differential field F_i, then it holds for the differential field F_{i+1}. This is somewhat lengthy, and is described in the following two sections.

1.3 The Problem of Integration

Suppose our integrand f lies in the differential field $F(\theta)$. Then according to the Liouville Structure Theorem,

$$\int f\,dx = y + \sum c_j \log h_j$$

where $y \in F(\theta), c_j \in \bar{K}$, and $b_j \in \bar{K}F(\theta)$.

Fraction Part. To compute the fraction part of y, let

$$f(\theta) = f_{poly}(\theta) + \sum_{i,j} \frac{f_{ij}(\theta)}{q_i(\theta)^j}$$

be the partial fraction decomposition of f. Now we make the following observation. Let $p(\theta)$ and $q(\theta)$ be relatively prime polynomials in θ (and such that $\theta \nmid q(\theta)$ if $\theta = e^\eta$). Then

$$\left(\frac{p(\theta)}{q(\theta)^k}\right)' = \frac{(p(\theta))'}{q(\theta)^k} - \frac{k\,p(\theta)\,(q(\theta))'}{q(\theta)^{k+1}}$$

Hence, if $a(\theta)$ is a polynomial in θ, we have the rule

$$\int \frac{a(\theta)}{q(\theta)^k}\,dx = \frac{b(\theta)}{q(\theta)^{k-1}} + \int \frac{c(\theta)}{q(\theta)^{k-1}}\,dx$$

where polynomial $b(\theta)$ is the solution to the equation

$$-(k-1)(q(\theta))'b(\theta) = a(\theta) \pmod{q(\theta), \theta}$$

which then determines the polynomial $c(\theta)$.

Each fraction

$$\sum_j \frac{f_{ij}(\theta)}{q_i(\theta)^j}$$

will contribute to the fraction part of y as follows. Let $q(\theta) = q_i(\theta)$ be the denominator appearing in such a fraction. We use the above rule to successively reduce the fraction to a fraction of the form

$$\frac{f_{i1}(\theta)}{q_i(\theta)}$$

Each $b(\theta)/q(\theta)^{k-1}$ generated this way becomes part of the fraction part of y.

Polynomial Part. After the fraction part of y is computed, our integration problem is reduced to a new integration problem

$$\int f\,dx = y + \sum c_j \log h_j$$

where $f(\theta) \in F(\theta)$ is of the form

$$f(\theta) = f_{poly}(\theta) + \sum \frac{f_{i1}(\theta)}{q_i(\theta)}$$

and where $y(\theta) \in F(\theta)$ is a polynomial in θ, $c_j \in \bar{K}$, and $h_j \in \bar{K}F(\theta)$. Suppose

$$y(\theta) = \sum y_i\,\theta^i = y_\alpha\,\theta^\alpha + \ldots$$
$$f_{poly}(\theta) = \sum f_i\,\theta^i = f_\beta\,\theta^\beta + \ldots$$

Differentiating the integral and ignoring the log part, we get

$$f(\theta) = f_\beta\,\theta^\beta + \ldots = (y(\theta) + \ldots)' = (y_\alpha\,\theta^\alpha + y_{\alpha-1}\,\theta^{\alpha-1} + \ldots)' + \ldots$$

We proceed to calculate the polynomial part of y according to whether $\theta = x, \log\eta$, or e^η.

Case $\theta = x$. In this case, we get the equation

$$\alpha\,y_\alpha\,\theta^{\alpha-1} + \ldots = f_\beta\,\theta^\beta + \ldots$$

from which we conclude

$$\alpha = \beta + 1$$
$$y_\alpha = \frac{f_\beta}{\alpha}$$

Case $\theta = \log\eta$. Assume y_α is a constant. Then we get the equation

$$\left(\alpha\,y_\alpha\,\theta' + y'_{\alpha-1}\right)\,\theta^{\alpha-1} = f_\beta\,\theta^\beta$$

from which we conclude

$$\alpha = \beta + 1$$
$$\int f_\beta\,dx = \alpha\,y_\alpha\,\theta + y_{\alpha-1} + C$$

Both sides of this equation are elements of $F(\theta)$ and are linear polynomials in θ. The integral is calculated via the induction hypothesis, and then examination of the coefficients of θ^1 and θ^0 in this equation completely determines y_α and determines $y_{\alpha-1}$ to within a constant. If we set

$$f_{new} = f - (y_\alpha\,\theta^\alpha + (y_{\alpha-1} + C)\,\theta^{\alpha-1})'$$

then f_{new} is a new polynomial in θ with $\deg(f_{new}, \theta) < \deg(f, \theta)$ and

$$\int f\,dx = y_\alpha\,\theta^\alpha + (y_{\alpha-1} + C)\,\theta^{\alpha-1} + \int f_{new}\,dx$$

We proceed to find the rest of the polynomial part y by integrating f_{new}.

Case $\theta = e^\eta$. In this case, we get the equation

$$(y'_\alpha + \alpha\,\eta'\,y_\alpha)\;\theta^\alpha + \ldots = f_\beta\,\theta^\beta + \ldots$$

from which we conclude

$$\alpha = \beta$$

$$y'_\alpha + \alpha\,\eta'\,y_\alpha = f_\beta$$

Now y_α, $\alpha\,\eta'$, and $f_\beta \in F$, so this last equation is a Risch differential equation over F, and we can use the induction hypothesis to solve for y_α.

All the y_i coefficients of the positive half and the negative half of the polynomial part y can be solved for in similar fashion.

Log Part. The log part of the integral is determined as follows. Suppose that

$$f(\theta) = \frac{p(\theta)}{q(\theta)}$$

where $p(\theta)$ and $q(\theta)$ are polynomials. We try to write

$$\int f(\theta)\,dx = y_0 + \sum c_i \log h_i(\theta)$$

Differentiate this equation to get

$$\frac{p(\theta)}{q(\theta)} = y_0{}' + \sum c_i\,\frac{(h_i(\theta))'}{h_i(\theta)}$$

Now we try to arrange that

$$q(\theta) = \prod h_i(\theta)$$

and this then gives us the following congruences:

$$\sum_i \left[c_i\,(h_i(\theta))' \prod_{j\neq i} h_j(\theta)\right] = p(\theta) \pmod{q(\theta), \theta}$$

$$c_i\,(h_i(\theta))' \prod_{j\neq i} h_j(\theta) = p(\theta) \pmod{h_i(\theta), \theta}$$

$$c_i\,(q(\theta))' = p(\theta) \pmod{h_i(\theta), \theta}$$

Hence,

$$\gcd(p(\theta) - c_i\,(q(\theta))', q(\theta)) = h_i(\theta)$$

The above results should at least motivate taking the following approach to compute the log part of the integral. Let

$$r(z) = \operatorname{res}(p(\theta) - z\,(q(\theta))', q(\theta), \theta)$$

The log part can be considered to be the sum of two sums,

$$\sum c_i \log h_i(\theta) = \sum_{h_i(\theta)\notin F} c_i \log h_i(\theta) + \sum_{h_i(\theta)\in F} c_i \log h_i(\theta)$$

the first sum including those $h_i(\theta)$ that depend on θ and the second sum including those $h_i(\theta)$ that are free of θ. We set

$$\sum_{h_i(\theta)\notin F} c_i \log h_i(\theta) = \sum_{r(c_i)=0} c_i \log(\gcd(p(\theta) - c_i\,(q(\theta))', q(\theta))$$

where the sum on the right-hand side is taken over the roots of the equation $r(z) = 0$. This method of using resultants to compute the log part of an integral is due to Trager.

Zero Part. We are at last left with an integral of the form

$$\int f\,dx = y_0 + \sum_{h_i(\theta)\in F} c_i \log h_i(\theta)$$

At this point, f, y_0, and the remaining $h_i(\theta)$ are free of θ and thus lie in the differential field F. To carry out this remaining integration, we use the induction hypothesis.

1.4 The Risch Differential Equation

Suppose we are working in a differential field $F(\theta)$. The Risch differential equation problem is to solve the equation

$$y' + f\,y = g$$

for $y \in F(\theta)$ where f and $g \in F(\theta)$. Assuming $F(\theta)$ is a reduced and normalized differential field, the solution y is unique and can be found as follows.

Fraction Part. To determine the fraction part of y, proceed as follows. Suppose $q(\theta)$ is a denominator in the partial fraction decomposition of f or g. If the q-adic expansions of y, f, and g are $y = \frac{y_\alpha}{q^\alpha} + \frac{y_{\alpha-1}}{q^{\alpha-1}} + \ldots f = \frac{f_\beta}{q^\beta} + \frac{f_{\beta-1}}{q^{\beta-1}} + \ldots g = \frac{g_\gamma}{q^\gamma} + \frac{g_{\gamma-1}}{q^{\gamma-1}} + \ldots$ then the Risch differential equation $y' + f\,y = g$

gives us the equation

$$\frac{-\alpha\,q'\,y_\alpha}{q^{\alpha+1}} + \ldots + \frac{f_\beta\,y_\alpha}{q^{\alpha+\beta}} + \ldots = \frac{g_\gamma}{q^\gamma} + \ldots$$

Hence, $\alpha = \gamma - \max(1, \beta)$. Determining y_α falls into three cases.

Case $\beta = 0$. In this case, $\alpha = \gamma - 1$ and

$$-\alpha\,(q(\theta))'\,y_\alpha(\theta) = g_\gamma(\theta) \pmod{q(\theta), \theta}$$

Case $\beta = 1$. In this case, $\alpha = \gamma - 1$ and

$$(-\alpha\,(q(\theta))' + f_\beta(\theta))\,y_\alpha(\theta) = g_\gamma(\theta) \pmod{q(\theta), \theta}$$

Case $\beta > 1$. In this case, $\alpha = \gamma - \beta$ and

$$f_\beta(\theta)\,y_\alpha(\theta) = g_\gamma(\theta) \pmod{q(\theta), \theta}$$

After y_α is determined, we set $y_{new} = y - y_\alpha/q^\alpha$ and form a new equation

$$y'_{new} + f\,y_{new} = g_{new}$$

and proceed to find the rest of the fraction part of y in similar fashion.

Polynomial Part. After the fraction part of y is computed, our original Risch differential equation is reduced to a new Risch differential equation of the form $y' + f\,y = g$ for some f and $g \in F(\theta)$, where $y(\theta)$ is known to be a polynomial in θ. Clear the denominators of f and g to obtain an equation

$$r\,y' + s\,y = t$$

where $r(\theta), s(\theta)$, and $t(\theta)$ are polynomials in θ. Let us suppose that the lead terms of $y(\theta), r(\theta), s(\theta)$, and $t(\theta)$ are given by

$$y(\theta) = y_\alpha\,\theta^\alpha + y_{\alpha+1}\,\theta^{\alpha+1} + \ldots$$
$$r(\theta) = r_\beta\,\theta^\beta + \ldots$$
$$s(\theta) = s_\gamma\,\theta^\gamma + \ldots$$
$$t(\theta) = t_\delta\,\theta^\delta + \ldots$$

Then the equation $r\,y' + s\,y = t$ becomes

$$(r_\beta\,\theta^\beta + \ldots)\,(y_\alpha\,\theta^\alpha + \ldots)' + (s_\gamma\,\theta^\gamma + \ldots)\,(y_\alpha\,\theta^\alpha + \ldots) = (t_\delta\,\theta^\delta + \ldots)$$

We proceed according to whether $\theta = x, \log\eta$, or e^η.

Polynomial in $\theta = x$. The equation $r\,y' + s\,y = t$ becomes

$$\alpha\, r_\beta\, y_\alpha\, \theta^{\alpha+\beta-1} + \ldots + s_\gamma\, y_\alpha\, \theta^{\alpha+\gamma} + \ldots = t_\delta\, \theta^\delta + \ldots$$

Case $\beta - 1 > \gamma$. In this case, $\alpha = \max(\delta - \beta + 1, 0)$ and

$$y_\alpha = \begin{cases} t_\delta/(\alpha\, r_\beta) & \alpha > 0 \\ t_\delta/s_\gamma & \alpha = 0 \end{cases}$$

Case $\beta - 1 = \gamma$. In this case, $\alpha = \delta - \gamma$ and

$$y_\alpha = \frac{t_\delta}{\alpha\, r_\beta + s_\gamma}$$

Case $\beta - 1 < \gamma$. In this case, $\alpha = \delta - \beta$ and $y_\alpha = t_\delta/s_\gamma$

Polynomial in $\theta = \log\eta$. Assume y_α is a constant. The equation $r\,y' + s\,y = t$ takes one of the following three forms: (1) $\quad r_\beta\,\left(\alpha\, y_\alpha\, \theta' + y'_{\alpha-1}\right)\, \theta^{\alpha+\beta-1} + \ldots + s_\gamma\, y_\alpha\, \theta^{\alpha+\gamma} + \ldots = t_\delta\, \theta^\delta + \ldots \qquad (y_\alpha \neq 0, \alpha > 0)$

(2) $\quad r_\beta\, y'_{\alpha-1}\, \theta^{\alpha+\beta-1} + \ldots + s_\gamma\, y_{\alpha-1}\, \theta^{\alpha+\gamma-1} + \ldots = t_\delta\, \theta^\delta + \ldots \qquad (y_\alpha = 0, \alpha > 0)$

(3) $\quad s_\gamma\, y_0\, \theta^\gamma + \ldots = t_\delta\, \theta^\delta + \ldots \qquad (\alpha = 0)$ **Case $\beta - 1 > \gamma$.** If $\delta - \beta + 1 > 0$, then equation (1) or (2) applies, $\alpha = \delta - \beta + 1$, and

$$r_\beta\,\left(\alpha\, y_\alpha\, \theta' + y'_{\alpha-1}\right) = t_\delta$$

Divide the last equation by r_β and integrate to get the equation

$$\int \frac{t_\delta}{r_\beta}\, dx = \alpha\, y_\alpha\, \theta + y_{\alpha-1} + C$$

Both sides of this equation are elements of $F(\theta)$ and are linear polynomials in θ. The integral is calculated via recursion (the induction hypothesis) and then examination of the coefficients of θ^1 and θ^0 in this equation completely determines y_α and determines $y_{\alpha-1}$ to within a constant. Set

$$y_{new} = y - y_\alpha\, \theta^\alpha - (y_{\alpha-1} + C)\, \theta^{\alpha-1}$$

and form a new equation

$$r\, y'_{new} + s\, y_{new} = t_{new}$$

and proceed to find the rest of the polynomial y in similar fashion.

If $\gamma = \delta$, then equation (3) applies, $\alpha = 0$, and $y_0 = t_\delta/s_\gamma$

Case $\beta - 1 = \gamma$. If $\delta - \gamma > 0$, then equation (1) applies, $\alpha = \delta - \gamma$, and

$$r_\beta\,\left(\alpha\, y_\alpha\, \theta' + y'_{\alpha-1}\right) + s_\gamma\, y_\alpha = t_\delta$$

Divide the last equation by r_β and integrate to get the equation

$$\alpha\, y_\alpha \theta + y_{\alpha-1} + y_\alpha\, \left(\int \frac{s_\gamma}{r_\beta}\, dx\right) = \int \frac{t_\delta}{r_\beta}\, dx$$

The integrals in this equation must be of the form $\displaystyle\int \frac{s_\gamma}{r_\beta}\, dx = a_1\, \theta + a_0 \int \frac{t_\delta}{r_\beta}\, dx = b_1\, \theta + b_0$ Compute these two

integrals to establish the coefficients a_0, a_1, b_0, and b_1. The quantities $a_1\, \theta + a_0$ and $b_1\, \theta + b_0$ are then substituted into the Risch differential equation. Examination of the coefficients of θ^1 and θ^0 in this equation completely determines y_α and determines $y_{\alpha-1}$ to within a constant. If $\alpha + b_1 \neq 0$ then $y_\alpha = \dfrac{a_1}{\alpha + b_1} y_{\alpha-1} = a_0 - b_0\, y_\alpha + C$

If $\alpha + b_1 = 0$ then $y_\alpha = 0\; y_{\alpha-1} = b_0 + C$ As in the previous case, these results yield a new Risch differential

equation from which the rest of the polynomial y is determined.

If $\delta - \gamma = 0$, then equation (3) applies, $\alpha = 0$, and $y_\alpha = t_\delta / s_\gamma$

Case $\beta = \gamma$. In this case, we don't assume y_α constant, but compute the lead coefficient y_α, whatever it happens to be. From

$$r_\beta\, y'_\alpha\, \theta^{\alpha+\beta} + \ldots + s_\gamma\, y_\alpha\, \theta^{\alpha+\gamma} + \ldots = s_\delta\, \theta^\delta + \ldots$$

we get $\alpha = \delta - \gamma$ and

$$r_\beta\, y'_\alpha + s_\gamma\, y_\alpha = t_\delta$$

which is a Risch differential equation for y_α that can be solved by recursion (the induction hypothesis).

Case $\beta < \gamma$. Like the previous case, we determine the lead coefficient y_α of y. We get $\alpha = \delta - \gamma$ and $y_\alpha = t_\delta / s_\gamma$.

Polynomial in $\theta = e^\eta$. The equation $r\, y' + s\, y = t$ becomes

$$r_\beta\, \left(y'_\alpha + \alpha\, \eta'\, y_\alpha\right)\, \theta^{\alpha+\beta} + \ldots + s_\gamma\, y_\alpha\, \theta^{\alpha+\gamma} + \ldots = t_\delta\, \theta^\delta + \ldots$$

Case $\beta > \gamma$. In this case, $\alpha = \delta - \beta$ and

$$r_\beta\, y'_\alpha + \alpha\, \eta'\, r_\beta\, y_\alpha = t_\delta$$

which is a Risch differential equation for y_α that can be solved by recursion (the induction hypothesis).

Case $\beta = \gamma$. In this case, $\alpha = \delta - \beta$ and

$$r_\beta\, y'_\alpha + \left(\alpha\, \eta'\, r_\beta + s_\gamma\right)\, y_\alpha = t_\delta$$

which is a Risch differential equation for y_α that can be solved by recursion (the induction hypothesis).

Case $\beta < \gamma$. In this case, $\alpha = \delta - \gamma$ and $y_\alpha = t_\delta / s_\gamma$

After $\alpha > 0$ and y_α are determined according to one of the three cases above, we set $y_{new} = y - y_\alpha\, \theta^\alpha$ to get a new equation

$$r\, y'_{new} + s\, y_{new} = t_{new}$$

and proceed to determine the rest of the positive half of y in similar fashion.

To determine the negative half of y, we view y, r, s, t as polynomials in the "variable" θ^{-1} and proceed as above for the positive half of y.

Finally, to determine the zero part of y, let f_0 and $g_0 \in F$ be the coefficients of θ^0 in the series expansions of $f = s/r$ and $g = t/r$ with respect to the variable θ:

$$f = \frac{s}{r} = \ldots + f_1\, \theta + f_0 + f_{-1}\, \theta^{-1} + \ldots$$

$$g = \frac{t}{r} = \ldots + g_1\, \theta + g_0 + g_{-1}\, \theta^{-1} + \ldots$$

Then we obtain the Risch differential equation

$$y_0{}' + f_0\, y_0 = g_0$$

which we solve via recursion (the induction hypothesis).

1.5 Algebraics

Most of the work to date on integration of expressions containing algebraics has been carried out by Bronstein (1990a), Davenport (1981), and Trager (1984). This work is very complicated, and only a small part of it has been implemented in *Mathematica* so far. Treatment of simple radical extensions, unnested radicals, is one case that has been implemented.

Suppose we are working in a differential field $F(\phi, \theta)$ where $\theta = \sqrt[n]{r}$ is a simple radical extension of a pure transcendental field $F(\phi)$ and r depends on ϕ. We wish to compute the integral

$$\int f\, \theta\, dx = y\, \theta + \sum c_i\, \log h_i$$

where $f, y \in F(\phi)$, $c_i \in \bar{K}$, and $h_i \in \bar{K}F(\phi, \theta)$.

Fraction Part. To compute the fraction part of y, write

$$f = \frac{a}{u\,v^k}$$

where $a(\phi), u(\phi), v(\phi) \in F(\phi)$ are polynomials in ϕ such that $u\,v^k = \mathrm{Den}(f)$ is a square-free factorization of the denominator of f with

$$\mathrm{Den}(f) = \prod_{i=1}^{k} q_i(\phi)^i$$
$$v = q_k(\phi)$$
$$u = \mathrm{Den}(f)/v^k$$

We try to write

$$\int \frac{a}{u\,v^k}\,\theta\,dx = \frac{b}{v^{k-1}}\,\theta + \int \frac{c}{u\,v^{k-1}}\,\theta\,dx$$

Suppose that

$$\theta' = \frac{m}{e}\,\theta$$

where $m(\phi), e(\phi) \in F(\phi)$ are polynomials in ϕ. Then we must have

$$\frac{a}{u\,v^k}\,\theta = -\frac{(k-1)\,b\,v'}{v^k}\,\theta + \frac{b\,m}{v^{k-1}\,e}\,\theta$$

Next, arrange to have $e(\phi) \mid u(\phi)\,v(\phi)$ by setting

$$a_{new} = \frac{e}{\gcd(e, u\,v)}\,a$$
$$u_{new} = \frac{e}{\gcd(e, u\,v)}\,u$$

if necessary, so that $u\,v/e$ is a polynomial in ϕ. Then the unknown polynomial $b(\phi)$ is determined by the equation

$$\left(-(k-1)\;u\,v' + \frac{u\,v}{e}\,m\right)\,b = a \pmod{v, \theta}$$

After solving for $b(\phi)$, the polynomial $c(\phi)$ can be determined. We are then left with the integration problem

$$\int \frac{c}{u\,v^{k-1}}\,\theta\,dx$$

which is handled in similar fashion.

Polynomial Part. Computing the polynomial part of the above integral proceeds very similarly to computing the polynomial part of an integral in purely transcendental fields, but with the difference that instead of working with polynomials, it is necessary to work with Puiseux series expansions. A Puiseux series expansion is a generalization of a Laurent series expansion in which exponents are allowed to be rational numbers instead of just integers. There are three cases to consider, $\theta = x, \log\eta$, and e^{η}.

Log Part. This is perhaps the most difficult part of integrating expressions containing algebraics. The complete theory has yet to be implemented, but we do implement the following special cases.

In the case where r is the square root of a linear polynomial, we have

$$\int \frac{dx}{(e\,x+f)\;\sqrt{a\,x+b}} = \frac{-2}{e\sqrt{t}}\tanh^{-1}\left(\frac{\sqrt{a\,x+b}}{\sqrt{t}}\right) = \frac{2}{e\sqrt{-t}}\tan^{-1}\left(\frac{\sqrt{a\,x+b}}{\sqrt{-t}}\right)$$

where $t = b - a\,f/e$.

In the case where r is the square root of a quadratic polynomial, we have

$$\int \frac{dx}{\sqrt{a\,x^2+b\,x+c}} = L(a, 2\,a\,x+b, 2\,\sqrt{a\,x^2+b\,x+c})$$

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = -L(c, \frac{bx+2c}{x}, \frac{2\sqrt{ax^2+bx+c}}{x})$$
$$\int \frac{dx}{(ex+f)\sqrt{ax^2+bx+c}} = -\frac{1}{e}L(\alpha, \beta, \frac{(ex+f)\sqrt{ax^2+bx+c}}{2e})$$

where α and β and L, which is a logarithm, are given by

$$\alpha = \frac{ce^2 - bef + af^2}{e^2}$$
$$\beta = \frac{2ce - bf + bex - 2afx}{ex+f}$$
$$L(u, v, w) = \frac{1}{\sqrt{u}} \log(\frac{v}{\sqrt{u}} + w)$$

In some circumstances, it is possible to write the $L(u, v, w)$ logarithm as an inverse trigonometric function, $\sinh^{-1}$ or $\sin^{-1}$.

1.6 Special Functions

There have been some efforts to generalize the Risch algorithm to handle special functions. We mention the work on log integrals, exponential integrals, and error functions by Cherry (1983) and Cherry (1989) here. The general idea is that at the point in the Risch algorithm where we must compute the log part of an integral, we allow instances of *li*, *ei*, and *erf* in addition to new logarithms.

Log Integrals. Suppose we are working in a differential field $F(\phi, \theta)$ where $\theta = \log \eta$. Then if $p_0, q_0, q_1 \in F(\phi)$, we try to write

$$\int \frac{p_0}{q_1\theta + q_0}\,dx = y_0 \log(\log u + c) + \sum y_i \mathrm{li}((e^c u)^i)$$

where $\log u \in F(\phi, \theta)$ and c and y_i are constants. We must have

$$\theta' + \left(\frac{q_0}{q_1}\right)' = \sum_{i\in E} \rho_i \eta_i' + \sum_{i\in L} \frac{\rho_i \eta_i'}{\eta_i}$$
$$u = \left(\prod_{i\in E} (e^{\eta_i})^{\rho_i} \prod_{i\in L} {\eta_i}^{\rho_i}\right)^{1/\rho}$$

where ρ_i are rationals and

$$\rho = \gcd(\rho_i)$$

The constant c is given by

$$c = \frac{1}{\rho}\left(\theta + \frac{q_0}{q_1}\right) - \log u$$

Now let

$$\sigma = \frac{p_0 u}{\rho q_1 u'} = y_0 + \sum i\, y_i e^{ic} u^i$$

We determine the highest power α of u appearing on the right-hand side of this equation as follows. If the series expansions of u and σ with respect to ϕ are

$$u = u_\beta \phi^\beta + \ldots$$
$$\sigma = \sigma_\gamma \phi^\gamma + \ldots$$

then comparing leading terms of these series expansions gives the equation

$$y_\alpha e^{\alpha c} {u_\beta}^\alpha \phi^{\alpha\beta} = \sigma_\gamma \phi^\gamma$$

from which we conclude

$$\alpha = \frac{\gamma}{\beta}$$

$$y_\alpha = \begin{cases} \sigma_\gamma / (e^{\alpha c} {u_\beta}^\alpha) & \alpha > 0 \\ \sigma_0 & \alpha = 0 \end{cases}$$

At this point, we can subtract $y_\alpha\, e^{\alpha\, c}\, u^\alpha$ from σ and then go on to determine the rest of the y_i in a similar manner. (However, in the case $\beta = 0$ more complicated tactics must be used.)

Exponential Integrals. Suppose we are working in a differential field $F(\phi, \theta)$ where $\theta = e^\eta$. Then if $f \in F(\phi)$ and η both depend on ϕ, we try to write

$$\int f\,\theta\, dx = \sum \frac{y_i ei(\eta + c_i)}{e^{c_i}} + \frac{\sqrt{\pi}}{2} \sum \frac{y_i \operatorname{erf}(\sqrt{-\eta - c_i})}{e^{c_i}}$$

where c_i and y_i are constants. It is assumed that the rational part of this integral, if any, has already been subtracted away, and we are only trying to compute the log part now. Differentiating both sides of this equation and dividing by θ gives

$$f = \sum \frac{y_i \eta'}{\eta + c_i} - \sum \frac{y_i \eta'}{2\sqrt{-\eta - c_i}}$$

First we compute the part of the integral that gives exponential integrals. Then we compute the part of the integral that gives error functions. The two steps for computing the exponential integrals are described below. Suppose

$$\eta = \frac{p}{q}$$

There can be at most one c_i, call it c_0, such that $p + c_i\, q$ is independent of ϕ.

Step 1. Case $c_i \neq c_0$. Assume

$$f = \frac{u}{v_1 v_2}$$

where

$$v_1 = \text{product of factors of } v \text{ that do not divide } q$$
$$v_2 = \text{product of factors of } v \text{ that do divide } q$$

If we let

$$r(z) = \operatorname{res}(p + z\,q, v_1, \phi)$$

$$R = \{c \mid r(c) = 0\}$$

then we must have

$$f = \sum_{c_i \in R} \frac{y_i \eta'}{\eta + c_i} + \dots$$

for some constants y_i. Comparing numerators of fractions on both sides of the equation with denominators $p + c_i\, q$ (which all depend on ϕ and are relatively prime) allows us to determine these y_i.

Step 2. Case $c_i = c_0$. The one c_i, which we call c_0, such that $p + c_i\, q$ is independent of ϕ can only occur in one of two mutually exclusive situations. Either (1) p is independent of ϕ and $c_0 = 0$ or (2) $\deg(p, \phi) = \deg(q, \phi)$ and $c_0 = -\operatorname{lc}(p, \phi)/\operatorname{lc}(q, \phi)$. In either case, we must have

$$f = -\frac{y_0 q'}{q} + \dots$$

and we compute y_0 by comparing numerators of fractions on both sides of the equation with denominator q.

Error Functions. After the exponential integrals have been calculated for

$$\int f\,\theta\, dx$$

where $\theta = e^\eta$ and $f, \eta \in F(\phi)$, we calculate the error functions. Again, we suppose

$$\eta = \frac{p}{q}$$

We try to write

$$f = \sum y_i \left(\frac{r_i}{s}\right)'$$

where

$$\left(\frac{r_i}{s}\right)^2 = \eta + c_i$$

Compute s such that $s^2 = q$. Then

$$\frac{2f}{s(p/q)'} = \sum \frac{y_i}{r_i}$$

We let

$$h = \mathrm{Den}(\frac{2f}{s(p/q)'})$$

Step 1. Assume just one error function. Then $ch = r_1$ for some c independent of ϕ. Since h and q are relatively prime, quantities c and c_1 are uniquely determined from the equation

$$p = c^2h^2 - c_1q$$

if they exist. Then $r_1 = ch$ and

$$y_1 = f/\left(\frac{r_1}{s}\right)'$$

If this step fails, then we go on to Step 2.

Step 2. Assume two error functions. Then $ch = r_1r_2$ and

$$p^2 = c^2h^2 - ((c_1 + c_2)\,p + c_1c_2q)\,q$$

Since h and q are relatively prime and p and q are relatively prime, c and constants a and b are uniquely determined from the equation

$$p^2 = c^2h^2 - (ap + bq)\,q$$

Then c_1, c_2 are roots of the quadratic $z^2 - az + b$. Calculate r_1, r_2 from $r_i^2 = p + c_iq$ and y_1, y_2 from

$$c\,\mathrm{Num}(\frac{2f}{s(p/q)'}) = y_1r_2 + y_2r_1$$

1.7 Undefined Functions

Here we are concerned with how to integrate expressions such as

$$\frac{u'(x)}{v(x)} - \frac{u(x)\,v'(x)}{v(x)^2}$$

where u and v are undefined functions.

Description of the Problem. Suppose we are working in a differential field $F(\theta, \theta_i)$ where $\theta = u(\vec{g})$, $\theta_i = u_i(\vec{g})$, u_i = the ith partial derivative of undefined function u, and the ith component of $\vec{g}$ is the first component of $\vec{g}$ that is not constant. Then the derivative θ' is given by the chain rule to be

$$\theta' = \phi + \sigma$$

where

$$\phi = \theta_i\, g_i'$$

$$\sigma = \sum_{j>i} \theta_j\, g_j'$$

And we want to compute the integral

$$\int (f_0 + f_1\,\phi)\,dx$$

We try to reduce the integration problem in $F(\theta, \theta_i)$ to an integration problem in $F(\theta)$ by applying rules which will eliminate the part of the integrand which appears with ϕ.

Derivatives. The method for computing integrals of expressions involving undefined functions is suggested by examination of derivatives of expressions involving undefined functions. The derivative of a polynomial $p(\theta)$ is given by

$$(p(\theta))' = \bar{p}(\theta) + p(\theta)\phi$$

where

$$\bar{p}(\theta) = p(\theta) + p(\theta)\sigma$$

$$p(\theta) = \sum i p_i' \theta^{i-1}$$
$$p(\theta) = \sum p_i' \theta^i$$

And the derivative of a fraction is given by
$$\left(\frac{p}{q^k}\right)' = \frac{\bar{p}\, q - k\, p\, \bar{q}}{q^{k+1}} + \frac{p\, q - k\, p\, q}{q^{k+1}}\, \phi$$
Inspection of these results leads to the algorithm below.

Fraction Part. To compute the fraction part of the integral, the rule
$$\int \frac{a}{q^k}\, \phi\, dx = \frac{b}{q^{k-1}} + \int \left(-\frac{\bar{b}\, q - (k-1)\; b\, \bar{q}}{q^k} + \frac{c}{q^{k-1}}\, \phi\right) dx$$
is used. The polynomials b and c are found by solving
$$a = b\, q - (k-1)\; b\, q + c\, q$$
Successive applications of the above rule convert the integral into a new integral that only involves a polynomial part and a log part.

Polynomial Part. To compute the polynomial part of the integral, we begin by observing that
$$\begin{aligned}\left(\frac{f\, \theta^{k+1}}{k+1}\right)' &= \frac{f'\, \theta^{k+1}}{k+1} + f\, \theta^k\, \theta' \\ &= \left(\frac{f'\, \theta^{k+1}}{k+1} + f\, \theta^k\, \sigma\right) + f\, \theta^k\, \phi\end{aligned}$$
Hence, we can use the rule
$$\int f\, \theta^k\, \phi\, dx = \frac{f\, \theta^{k+1}}{k+1} - \int \left(\frac{f'}{k+1}\, \theta^{k+1} + f\, \theta^k\, \sigma\right) dx$$
to determine the polynomial part of the integral.

Log Part. The log part of the integral is determined as follows. Suppose that
$$f(\theta) = \frac{p(\theta)}{q(\theta)}$$
where $p(\theta)$ and $q(\theta)$ are polynomials. Then if we let
$$r(z) = \mathrm{res}(p(\theta) - z\, q(\theta), q(\theta), \theta)$$
we have
$$\int \frac{p(\theta)}{q(\theta)}\, \phi\, dx = \sum_{r(c_i)=0} c_i\, \log(\gcd(p(\theta) - c_i\, \bar{q}(\theta), q(\theta)))$$

2 Definite Integration

■ 2.1 Contour Integration

Cauchy's Residue Theorem. If f is analytic except for isolated singularities in a domain D and C is a simple closed curve in D, then
$$\oint_C f(z)\, dz = 2\, \pi\, i \sum \mathrm{Res}(f(z), z_i)$$

Residues. At a simple pole z_0 we have
$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$
and at a pole z_0 of order n we have
$$\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\partial^{n-1}}{\partial z^{n-1}} \left[(z - z_0)^n f(z)\right]$$

If the Laurent series expansion for $f(z)$ about z_0 is
$$f(z) = \frac{f_{-n}}{(z - z_0)^n} + \ldots + \frac{f_{-1}}{(z - z_0)} + f_0 + \ldots$$
then $\mathrm{Res}(f(z), z_0) = f_{-1}$.

Method 1. Suppose $R(x)$ is a rational function in x. Then we have
$$\begin{aligned}\int_{-\infty}^{\infty} R(x)\,dx &= \oint_{\text{upper half plane}} R(z)\,dz \\ &= 2\,\pi\,i \sum_{\mathrm{Re}(z_i)>0} \mathrm{Res}(R(z), z_i)\end{aligned}$$
the sum being over the set of poles in the upper half plane.

Method 2. Again suppose $R(x)$ is a rational function in x. Then we have a method to calculate definite integrals of trigonometric expressions as follows:
$$\begin{aligned}\int_0^{2\pi} R(e^{i\,\theta})\,d\theta &= \oint_{|z|=1} R(z)\,\frac{dz}{i\,z} \\ &= 2\,\pi\,i \sum_{|z_i|<1} \mathrm{Res}(R(z), z_i)\end{aligned}$$
the sum being over the set of poles occuring inside the unit circle.

Method 3. Suppose $R(x)$ is a rational function in x and $\alpha > 0$. Then we have
$$\int_{-\infty}^{\infty} R(x)\,e^{i\,\alpha\,x}\,dx = 2\,\pi\,i \sum_{\mathrm{Re}(z_i)>0} \mathrm{Res}(R(z)\,e^{i\,\alpha\,z}, z_i)$$
For $\alpha < 0$, make the change of variable $x \mapsto -x$.

Method 4. If $f(x)$ is an even function, then
$$\int_0^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$$
and we can try to apply Method 1 to the right-hand side integral.

■ 2.2 Meijer's G Function

Meijer's G function is a generalization of the hypergeometric ${}_pF_q$ function and is defined by the Mellin-Barnes type contour integral
$$G(\vec{a}; \vec{b}; \vec{c}; \vec{d}; x) = \frac{1}{2\,\pi\,i} \oint_L \frac{\prod \Gamma(1 - a_i + s) \prod \Gamma(c_i - s)}{\prod \Gamma(b_i - s) \prod \Gamma(1 - d_i + s)}\,x^s\,ds$$
where L is a contour that embraces the poles of $\Gamma(1 - a_i + s)$. Meijer's G function is useful because it allows us to evaluate a wide class of definite integrals involving hypergeometric type functions.

Properties. Some properties of Meijer's G function:
$$\begin{aligned}x^p\,G(\vec{a}; \vec{b}; \vec{c}; \vec{d}; \alpha\,x) &= \alpha^{-p}\,G(\vec{a} + p; \vec{b} + p; \vec{c} + p; \vec{d} + p; \alpha\,x) \\ G(u, \vec{a}; \vec{b}; \vec{c}; u, \vec{d}; x) &= G(\vec{a}; \vec{b}; \vec{c}; \vec{d}; x) \\ G(\vec{a}; u, \vec{b}; \vec{c}; u, \vec{d}; x) &= G(\vec{a}; \vec{b}; \vec{c}; \vec{d}; x)\end{aligned}$$
$$\frac{\partial}{\partial x} G(u, \vec{a}; \vec{b}; \vec{c}; \vec{d}; x) = x^{-1}\,G(u - 1, \vec{a}; \vec{b}; \vec{c}; \vec{d}; x) + (u - 1)\,x^{-1}\,G(u, \vec{a}; \vec{b}; \vec{c}; \vec{d}; x)$$
$$\frac{\partial}{\partial x} G(\vec{a}; \vec{b}; u, \vec{c}; \vec{d}; x) = -x\,G(\vec{a}; \vec{b}; u - 1, \vec{c}; \vec{d}; x) + u\,x\,G(\vec{a}; \vec{b}; u, \vec{c}; \vec{d}; x)$$

Examples. A considerable inventory of functions can be written in terms of Meijer's G function.
$$(1 + x)^p = \frac{1}{\Gamma(-p)}\,G(1 + p; ; 0; ; x)$$

$$e^{-x} = G(;;0;;x)$$
$$\log(1+x) = G(1,1;;1;0;x)$$
$$\sin(x) = \sqrt{\pi}\, G(;;\frac{1}{2};0;\frac{x^2}{4})$$
$$\cos(x) = \sqrt{\pi}\, G(;;0;\frac{1}{2};\frac{x^2}{4})$$
$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}}\, G(1;;\frac{1}{2};0;x^2)$$
$$J_\nu(x) = G(;;\frac{\nu}{2};-\frac{\nu}{2};\frac{x^2}{4})$$
$$K_\nu(x) = G(;;\frac{\nu}{2},-\frac{\nu}{2};;\frac{x^2}{4})$$
$$N_\nu(x) = G(;-\frac{\nu}{2}-\frac{1}{2};-\frac{\nu}{2},\frac{\nu}{2};-\frac{\nu}{2}-\frac{1}{2};\frac{x^2}{4})$$
$$H_\nu(x) = G(\frac{\nu}{2}+\frac{1}{2};;\frac{\nu}{2}+\frac{1}{2};-\frac{\nu}{2},\frac{\nu}{2};\frac{x^2}{4})$$
$$J_\mu(x)\, J_\nu(x) = \frac{1}{\sqrt{\pi}}\, G(\frac{1}{2},0;;\frac{\mu+\nu}{2};\frac{\nu-\mu}{2},\frac{\mu-\nu}{2},\frac{-\mu-\nu}{2};x^2)$$
$$K_\mu(x)\, K_\nu(x) = \frac{1}{2\sqrt{\pi}}\, G(;0,\frac{1}{2};\frac{\mu+\nu}{2},\frac{\nu-\mu}{2},\frac{\mu-\nu}{2},\frac{-\mu-\nu}{2};;x^2)$$
$${}_2F_1(a,b;c;-x) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\, G(1-a,1-b;;0;1-c;x)$$
$${}_pF_q(\vec{a};\vec{b};-x) = \frac{\prod \Gamma(b_i)}{\prod \Gamma(a_i)}\, G(1-\vec{a};;0;1-\vec{b};x)$$

Also, the Heavyside function

$$H(1-x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

can be represented as

$$H(1-x) = G(;1;;0;x)$$

Particular Cases of Meijer's G Function There are also rules for converting out of Meijer's G notation back into more familiar functions.

$$G(;;a;b;x) = x^{(a+b)/2}\, J_{a-b}(2\sqrt{x})$$
$$G(;;a,b;;x) = 2\, x^{(a+b)/2}\, K_{a-b}(2\sqrt{x})$$

See Erdélyi, *et al.* (1954) and Prudnikov, *et al.* (1990) for more such rules.

Converting Functions into Meijer G Notation. Here is an algorithm to convert MacLaurin series expansions of simple functions into hypergeometric ${}_pF_q$ notation, and from there into Meijer G notation. First, assume we have a series expansion

$$y(x) = \sum y_n\, x^n$$

where the ratios y_{n+1}/y_n are rational functions in n. (If this doesn't happen, you will have to use a more sophisticated approach, or the task might actually be impossible.) If we can write

$$(n+1)\,\frac{y_{n+1}}{y_n} = c\,\frac{\prod(n+a_i)}{\prod(n+b_i)}$$

for some a_i, b_i, and c independent of n, then we have

$$\begin{aligned} y(x) &= y_0\, {}_pF_q(\vec{a};\vec{b};c\,x) \\ &= y_0\,\frac{\prod \Gamma(b_i)}{\prod \Gamma(a_i)}\, G(1-\vec{a};;0;1-\vec{b};-c\,x) \end{aligned}$$

Example. From the series expansion

$$\sin x = x \sum \frac{(-1)^n}{(2\,n+1)!}\,(x^2)^n$$

we deduce the following:

$$(n+1)\,\frac{y_{n+1}}{y_n} = (n+1)\,\frac{-1}{(2\,n+2)\,(2\,n+3)} = -\frac{1}{4}\cdot\frac{1}{(n+\frac{3}{2})}$$

and thus

$$\begin{aligned}\sin x &= x\,{}_0F_1(;\frac{3}{2};-\frac{x^2}{4})\\ &= x\,\Gamma(\frac{3}{2})\,G(;;0;-\frac{1}{2};\frac{x^2}{4})\\ &= \frac{1}{2}\sqrt{\pi}\,x\,G(;;0;-\frac{1}{2};\frac{x^2}{4})\\ &= \sqrt{\pi}\,G(;;\frac{1}{2};0;\frac{x^2}{4})\end{aligned}$$

where in the last step we absorb $x/2$ into the Meijer G function.

Definite Integrals. A definite integral of a single G function is a product of Γ functions

$$\int_0^\infty G(\vec{a};\vec{b};\vec{c};\vec{d};\alpha\,x)\,dx = \alpha^{-1}\,\frac{\prod\Gamma(-a_i)\prod\Gamma(c_i+1)}{\prod\Gamma(b_i+1)\prod\Gamma(-d_i)}$$

A definite integral of a product of two G functions is itself a G function.

$$\int_0^\infty x^p\,G(\vec{a};\vec{b};\vec{c};\vec{d};\alpha\,x^q)\,dx = \frac{1}{q}\,\alpha^{-\frac{p+1}{q}}\,\frac{\prod\Gamma(-a_i+1-\frac{p+1}{q})\prod\Gamma(c_i+\frac{p+1}{q})}{\prod\Gamma(b_i+\frac{p+1}{q})\prod\Gamma(-d_i+1+\frac{p+1}{q})}$$

$$\int_0^\infty G(\vec{a}_1;\vec{b}_1;\vec{c}_1;\vec{d}_1;\alpha\,x)\,G(\vec{a}_2;\vec{b}_2;\vec{c}_2;\vec{d}_2;\beta\,x)\,dx = \alpha^{-1}\,G(-\vec{c}_1,\vec{a}_2;-\vec{d}_1,\vec{b}_2;-\vec{a}_1,\vec{c}_2;-\vec{b}_1,\vec{d}_2;\frac{\beta}{\alpha})$$

$$\begin{aligned}&\int_0^\infty x^p\,G(\vec{a}_1;\vec{b}_1;\vec{c}_1;\vec{d}_1;\alpha\,x^q)\,G(\vec{a}_2;\vec{b}_2;\vec{c}_2;\vec{d}_2;\beta\,x^q)\,dx\\ &= \frac{1}{q}\,\alpha^{-\frac{p+1}{q}}\,G(-\vec{c}_1+1-\frac{p+1}{q},\vec{a}_2;-\vec{d}_1+1-\frac{p+1}{q},\vec{b}_2;\\ &\qquad -\vec{a}_1+1-\frac{p+1}{q},\vec{c}_2;-\vec{b}_1+1-\frac{p+1}{q},\vec{d}_2;\frac{\beta}{\alpha})\end{aligned}$$

2.3 Differentiation under the Integral Sign

It is possible to insert logarithms into the integrands of some of our definite integration rules by using a technique called differentiation under the integral sign. Here are some such rules:

Rule 1.

$$\int_0^\infty f(x)\,(\log x)^n\,dx = \left(\frac{\partial^n}{\partial z^n}\int_0^\infty x^z f(x)\,dx\right)\Bigg|_{z=0}$$

Rule 2.

$$\int_0^\infty f(x)\,(\log(x^p+a^p))^n\,dx = \left(\frac{\partial^n}{\partial z^n}\int_0^\infty (x^p+a^p)^z f(x)\,dx\right)\Bigg|_{z=0}$$

Rule 3.

$$\int_0^\infty G(\vec{a};\vec{b};\vec{c};\vec{d};\alpha\,x)\,(\log x)^n\,dx = \alpha^{-1}\left(\frac{\partial^n}{\partial z^n}\,\frac{\prod\Gamma(-a_i-z)\prod\Gamma(c_i+z+1)}{\prod\Gamma(b_i+z+1)\prod\Gamma(-d_i-z)}\right)\Bigg|_{z=0}$$

3 Miscellaneous Integrals

Some miscellaneous integrals can be treated by table lookup. Here are some integrals in that category.

Polylogarithms

$$\int \frac{\log(x)}{x-1}\,dx = -\mathrm{Li}_2(1-x)$$

$$\int \frac{\log(x)}{1-x}\,dx = \mathrm{Li}_2(1-x)$$

$$\int \frac{\log(1-x)}{x}\,dx = -\mathrm{Li}_2(x)$$

$$\int \frac{\log(a\,x+b)}{x}\,dx = \log(a\,x+b)\,\log\left(-\frac{a\,x}{b}\right) + \mathrm{Li}_2\left(\frac{a\,x}{b}+1\right)$$

$$\int \frac{\log(x)}{a\,x+b}\,dx = \frac{1}{a}\,\log(x)\,\log(\frac{a\,x}{b}+1) + \frac{1}{a}\,\mathrm{Li}_2(-\frac{a\,x}{b})$$

$$\int \frac{\log(a\,x+b)}{c\,x+d}\,dx = \frac{1}{c}\,\log(a\,x+b)\,\log\left(\frac{a\,(c\,x+d)}{a\,d-b\,c}\right) + \frac{1}{c}\,\mathrm{Li}_2\left(\frac{c\,(a\,x+b)}{b\,c-a\,d}\right)$$

$$\int \frac{\mathrm{Li}_n(x)}{x}\,dx = \mathrm{Li}_{n+1}(x)$$

The Zeta Function. Gradshteyn and Ryzhik (1980) formulas 3.411 (1), (3), and (7) give us the following rules:

$$\int_0^\infty \frac{x^{\nu-1}}{e^{\mu\,x}-1}\,dx = \mu^{-\nu}\,\Gamma(\nu)\,\zeta(\nu)$$

$$\int_0^\infty \frac{x^{\nu-1}}{e^{\mu\,x}+1}\,dx = \mu^{-\nu}\,(1-2^{1-\nu})\,\Gamma(\nu)\,\zeta(\nu)$$

$$\int_0^\infty \frac{x^{\nu-1}\,e^{-\mu\,x}}{1-e^{-\beta\,x}}\,dx = \beta^{-\nu}\,\Gamma(\nu)\,\zeta(\nu,\frac{\mu}{\beta})$$

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