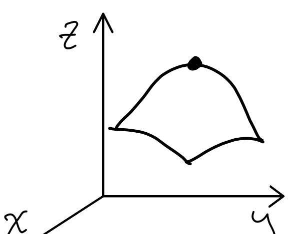
MAXIMUM AND MINIMUM VALUES (page 1 of 3)

Extreme Value Theorem - Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane, then there is at least one point in R where f attains a maximum value and at least one point in R where f attains a minimum value.

As in 2D, we have absolute extrema and relative extrema.
Let f be defined on a region R containing (x_0, y_0).

① f has a relative maximum at (x_0, y_0) if $f(x,y) \leq f(x_0, y_0)$ for all (x,y) in an open disk containing (x_0, y_0).
② f has a relative minimum at (x_0, y_0) if $f(x,y) \geq f(x_0, y_0)$ " "

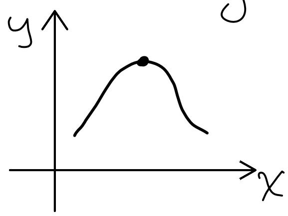


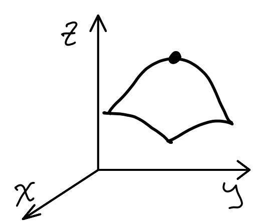
If f has a relative extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

A critical point of f is a point where the derivative $\rightarrow \vec{\nabla} f(x,y) =$ ____________ or ____________.

At a critical point (x_0, y_0) one of the following is true.
① $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
② $f_x(x_0, y_0)$ OR $f_y(x_0, y_0)$ does not exist.

In 2D, the tangent line at a relative max. or min. is ____________.
What do you think will happen in 3D?





Can we prove our conjecture?
Consider $\vec{\nabla} F(x,y,z)$ for $F(x,y,z) = f(x,y) - z = 0$.

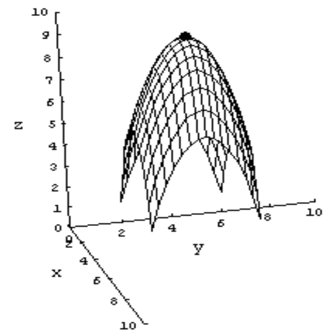
Example: $f(x,y) = -x^2 - y^2 + 4x + 8y - 11$

Find critical points: $f_x(x,y) =$

$f_y(x,y) =$

Set both $= 0$ and solve system.

Complete square to prove it's a maximum.



Second Partials Test

Let f have continuous partial derivatives on an open region containing a point (a,b) for which $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Define $D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$

① If $D > 0$ and $f_{xx}(a,b) > 0$, then f has a relative minimum at (a,b).

② If $D > 0$ and $f_{xx}(a,b) < 0$, then f has a relative maximum at (a,b).

③ If $D < 0$, then $(a,b,f(a,b))$ is a saddle point.

④ If $D = 0$, the test is inconclusive.

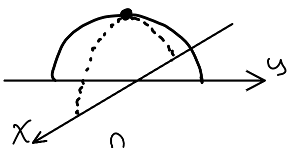
Note: $D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$

Above example: $f_{xx}(x,y) =$, $f_{yy}(x,y) =$, $f_{xy}(x,y) =$

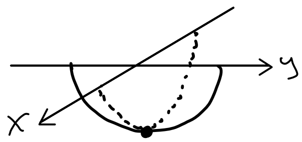
$f_{xx}(\ , \ ) =$, $f_{yy}(\ , \ ) =$, $f_{xy}(\ , \ ) =$

$D =$

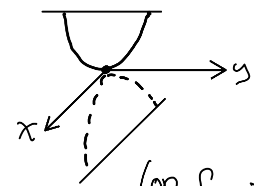
These conditions make sense. If you have a maximum or a minimum, then the curvatures (concavity) in both the x- and y-directions need to be in the same direction. At a saddle point, the x concavity and the y concavity are opposite.



$f_{xx} < 0$
$f_{yy} < 0$
$D = (-)(-) - (+) > 0$
Maximum



$f_{xx} > 0$
$f_{yy} > 0$
$D = (+)(+) - (+) > 0$
Minimum



$f_{xx} < 0$
$f_{yy} > 0$
(OR $f_{xx} > 0$ and $f_{yy} < 0$)
$D = (-)(+) - (+) < 0$
Saddle

Example: $f(x,y) = y^3 - 3x^2y - 3y^2 - 3x^2 + 1$

Note: To find absolute extrema, also consider values along the boundary of the given domain. This is analogous to checking the endpoints of your interval in two dimensions.