Parametric Surfaces with Mathematica

Don't Forget: Use **Shift-Enter** to run your code and *Mathematica* is **case-sensitive**.
To make the 3D graphics interactive, type and execute `<<RealTime3D`` (backwards apostrophe). This will let you rotate the 3D graphs and if you press Ctrl while clicking and dragging it will zoom.
To put it back in the default mode so you can see coordinate axes, type and execute `<<Default3D``

Many interesting surfaces can be written as $z = f(x, y)$ and graphed with the Plot3D command. But many times we need surfaces that are not functions of x and y. Just like we can use parametric equations to graph curves in space, we can use them to create three-dimensional surfaces. Curves use one parameter, t, to draw one-dimensional objects in three-dimensional space. Surfaces use two parameters, u and v, to draw two-dimensional objects in three-dimensional space.

The Basics

- For example, graph the surface $z = x\cos y$ for $-1 \le x \le 1,\ 0 \le y \le 2\pi$ using the code below:
  ```
  Plot3D[x*Cos[y],{x,-1,1},{y,0,2Pi}]
  ```
- To parametrize this surface, let $x = u$ and $y = v$. Then $z = u \cos v$ and $-1 \le u \le 1,\ 0 \le v \le 2\pi$.
  Graph the parametric surface using the code below and compare the two graphs:
  ```
  ParametricPlot3D[{u,v,u*Cos[v]},{u,-1,1},{v,0,2Pi}]
  ```
  Note that this is the same structure we use for graphing curves, but now with u <u>and</u> v, not just t.
- The general structure for graphing parametric surfaces is:
  ParametricPlot3D[{$x(u,v)$**,** $y(u,v)$**,** $z(u,v)$**},{u,**u_{min}**,**u_{max}**},{v,**v_{min}**,**v_{max}**}]**
- A common option is to remove the box around the plot. Insert `,Boxed->False` after the domain for v and inside the last `]`. Be sure you separate the domain from the option with a comma.

Practice with Simple Rectangular Surfaces

Suppose we want to graph the plane $x = 3$. That cannot be written as a function $z = f(x, y)$. We can use parametric equations to draw it. Use the code above, but fill in $x(u,v) = 3,\ y(u,v) = u,\ z(u,v) = v$ and choose your own domain for u and v. How would you graph $y = 3$? Try it.

- To graph both together, name each plot and use `Show`. After naming each use shift-enter again.
  ```
  xplane=ParametricPlot3D[ . . .
  yplane=ParametricPlot3D[ . . .
  Show[xplane, yplane]
  ```
  > You may find it useful to go to the last section of these directions and apply the example under "*Showing More than One Graph at a Time*."
- Can you draw a cube?
  Tip: For planes, you only need to plot two points in each dimension. Use `,PlotPoints->{2,2}` within the ParametricPlot3D command to graph 2 points in the u and v directions, respectively. Try using this option in one of the earlier plots. Why do you need more points for those graphs?

Practice with Cylindrical Coordinates

Let's use what we know about cylindrical coordinates to create some graphs.

- To graph the cylinder $x^2 + y^2 = 16$ use $x(u,v) = 4\cos u,\ y(u,v) = 4\sin u,\ z(u,v) = v$ with $0 \le u \le 2\pi,\ 0 \le v \le 5$. The radius is 4 and u acts like θ. Consider a new domain for u. Make a prediction and change your graph. (Remember to use `Cos[u]` for cos u, etc. and a space or * for multiplication.)
- What would you change to graph $x^2 + z^2 = 9$? Try it.
- Remember "Simple Cylindrical Surfaces"? Say z is constant, what do you get? Let's try $z = 5$ and allow r and θ to take on all possible values (well, a range of values). Let $r = u$ and $\theta = v$. Use: $x(u,v) = u\cos v,\ y(u,v) = u\sin v,\ z(u,v) = 5$ with $0 \le u \le 1,\ 0 \le v \le 2\pi$. Compare to cylindrical coords.
- Change the 5 to u and graph it. Now change it to v. Look at it from the top. Does it make sense?

Practice with Spherical Coordinates

Recall spherical coordinates: $x = \rho \sin\phi \cos\theta,\ \ y = \rho \sin\phi \sin\theta,\ \ z = \rho \cos\phi$.

- To graph a sphere of radius 5, use these conversion equations and replace ρ with 5 and let $\phi = u$ and $\theta = v$. For the domain use $0 \le u \le \pi,\ \ 0 \le v \le 2\pi$.
- Try changing the domain. Make predictions and check your ideas. Use **`Pi/2`**, for example.
- Graph the sphere again using the original domain. Now we'll try some "Simple Spherical Surfaces." To graph $\phi = \pi/4$ (constant), replace each u with **`Pi/4`** and change the 5's to the variable u.
- What would $\theta = \pi/4$ look like? Apply what you learned above to graph it. Is it what you expected?
- Parametric equations similar to spherical coordinates generate some interesting graphs. Try these:
  - $x(u,v) = u \sin u \cos v,\ \ y(u,v) = u \sin u \sin v,\ \ z(u,v) = u \cos u$ with $0 \le u \le \pi,\ \ 0 \le v \le \pi$
  - $x(u,v) = u \sin u \cos v,\ \ y(u,v) = v \sin u \sin v,\ \ z(u,v) = v \cos u$ with $0 \le u \le 2\pi,\ \ 0 \le v \le 3\pi/2$
  - $x(u,v) = 3 \sin u \cos 3v,\ \ y(u,v) = 3 \sin u \sin v,\ \ z(u,v) = 3 \cos u$ with $0 \le u \le \pi,\ \ 0 \le v \le 2\pi$
- Try some of your own variations.

Mini-Project

- Create your own three-dimensional image or images.
- Try to create something that uses multiple surfaces.
- Read the suggestions below about showing images together.
- Save your work and we'll share them later. (Please do NOT e-mail your files. They will fill up my inbox.) If your file is too large to save, delete the pictures and just save the code. You can always re-execute each command with shift-enter after re-opening the file.

Showing More than One Graph at a Time

Often we want more than one graph to show together, but we do not want to have to see all of the intermediate steps. You can suppress the output for a graph using the **`DisplayFunction`** option. Try the example below for two-dimensional graphs and see what happens. It works the same for 3D graphs.

Type the following as a single block of code. Use regular Enter to start each new line. After you type all three lines, use Shift-Enter.

```
myparabola=Plot[x^2,{x,-2,2}]
myline=Plot[5x-1,{x,-2,2}]
Show[myparabola,myline]
```

Now type the following as a single block of code or insert the **`DisplayFunction`** options as shown into the previous code. After you type all three lines, use Shift-Enter.

```
myparabola=Plot[x^2,{x,-2,2},DisplayFunction->Identity];
myline=Plot[5x-1,{x,-2,2},DisplayFunction->Identity];
Show[myparabola,myline,DisplayFunction->$DisplayFunction];
```

Compare how *Mathematica* drew the graphs and combined them in each case. The **`DisplayFunction`** option is very useful when creating complicated graphs and images.