Why *A New Kind of Science* Is Important for Mathematicians

by Todd Rowland

Introduction

Having worked with Stephen Wolfram in his Science Group for the past couple of years, starting shortly after completing my Ph.D. in mathematics at the University of Chicago, I have naturally been curious about how ***A New Kind of Science*** (NKS) would be received, especially by mathematicians. NKS is a very large and very unconventional book, and clearly presents a major challenge to those who would summarize or comment on it. I will address here just a few (chosen somewhat arbitrarily) of the many math-related ideas in NKS that I myself find intriguing but have not yet seen highlighted in the book's many reviews.

In a sense, NKS describes a new and potentially much more powerful way of mathematizing science than traditional approaches. But NKS also has profound implications for mathematics itself, not only via specific results in areas such as logic and discrete dynamical systems like cellular automata, but also in ways of thinking and methodologies. For instance, consider all possible axioms starting with the simplest examples, which is done on p. 804. It is apparent that there are a large number of possibilities that mathematics has not, until now, been investigating.

Instead of concentrating on systems that exhibit regularity, in NKS Wolfram has concentrated on the more general case of irregularity. The book's emphasis is on finding and exploring the simplest examples. NKS shows that advances in computational hardware and software over the last decade have made it surprisingly feasible to study such irregular systems systematically and make useful statements about them.

NKS contains many specific points of mathematical interest. A few examples: irreducible recursive functions (p. 130); a simple PDE with complex behavior (p. 166); the simplest nonperiodic tiling of finite type (p. 219); unequal circle packing examples (p. 350); space as a network (p. 476); network constraint systems (p. 483); generalized statistical tests (p. 597); functions representing nested patterns (p. 610); examples of the Post Correspondence Problem (p. 757); and axioms for mathematics (pp. 773-4).

However, instead of discussing these specifics, I have chosen here to examine the implications of the ideas of NKS for mathematics on a more general level.

What Is NKS?

NKS is a large book—800 pages of text written for a broad audience, with a minimum of technical language, plus more than 300 pages of technical notes containing some formulas, lots of computer code, and much historical background. The book is rich with examples, captured in numerous graphics that highlight essential details.

Superficially, the book may appear to belong to the popular science genre. Indeed it is intended to be accessible to the widest possible audience, but its many novel approaches and examples make it a primary document. The strategy of NKS and the results it presents are so different from traditional points of view that they would have been much more difficult to understand in another format.

Basic Motivation

A basic motivation behind NKS comes from the discovery that many systems defined by definite simple rules exhibit essentially unpredictable behavior. A good example is the rule 30 elementary cellular automaton. See Figure 1 for its evolution from the simplest nontrivial initial condition, a single black cell in an infinite array of white cells.

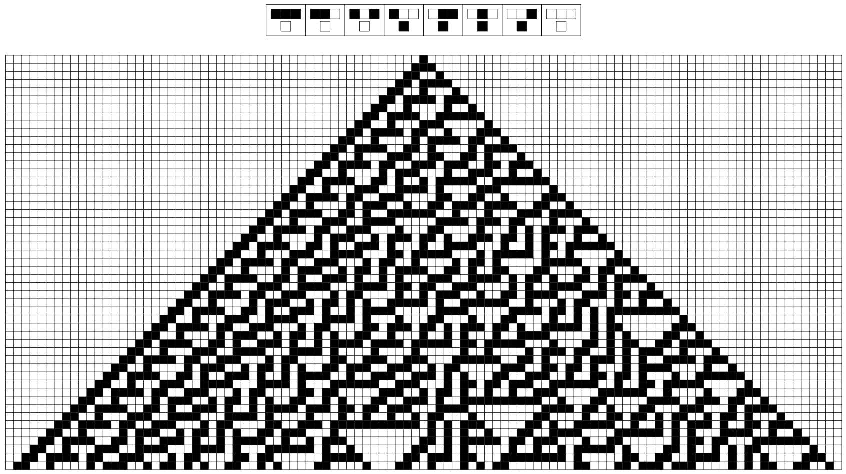

Figure 1

Many fundamental questions are raised by looking at this example:

• Is there some trick that will predict the future of rule 30, like a formula for the color at {x,t}? This question leads one to concepts like undecidability. More specifically, Wolfram explains that questions like this are computationally irreducible. The only way to find the answer is to evolve the system; there are no shortcuts.

• Is this irreducibility special in some way to rule 30? Wolfram says the answer is no. Part of his Principle of Computational Equivalence says that most natural systems are computationally irreducible. Another part of this principle says that the majority of natural systems are capable of universal computation, hence equivalent in their ultimate computational capabilites.

• From the computational point of view, is rule 30, then, a universal computer? No proof has yet been found, but some evidence is put forward in the Notes in NKS. The book gives an outline of a proof for rule 110. In general, proofs of universality have involved elaborate constructions. It would be interesting to find a practical test for universality.

•In what ways are the results random? Rule 30 is deterministic so the system evolved from it is not random in the sense that if we launch it with the same initial conditions, then the results will always be the same. But those results are random in the sense of being unpredictable. What Wolfram's experiments show is that these systems generate their own randomness, at least on an intuitive level, and that this phenomenon of intrinsic randomness generation may explain our perception of randomness in systems in the natural world. Moreover, he proposes a definition for randomness based on the NKS conjecture that the behavior of the natural world is based on simple rules.

To what extent can natural phenomena be demonstrated, explained, and studied by deterministic rule systems? This is the question that launched Wolfram's remarkable journey, leading him to discover the myriad systems described in NKS.

Mathematical Experimentation

Examples have always played a crucial role in the process of mathematics. They suggest which problems are of interest. They provide ideas for proofs. In recent years, computers have played a greater role, especially in number theory. Software, in particular *Mathematica*, has allowed a wide range of mathematics to be studied with the aid of computers.

NKS tells us a couple things about computer math experiments. First, the Principle of Computational Equivalence suggests that simple rules are more interesting than previously realized, because working with simplified versions of structures, and looking at what happens with them, can capture basic features of larger examples. So it makes sense to do a straightforward search for interesting examples of simple rules, and toexplore such examples in detail before moving on.

Moreover, Wolfram urges a greatly expanded role for experimentation in the study of basic objects of mathematics. He has found that the most obvious questions typically are undecidable; consequently, he believes that the best way to study them is through systematic experimentation. It is fairly clear that there are many unexplored possibilities for experimental mathematics.

Explaining Mathematical Behavior by Studying Simpler Systems

In NKS, Wolfram argues that before looking at the foundations of contemporary mathematics, it makes sense first to understand simpler systems, just as science has traditionally done in modeling the natural world.

To illustrate this, he considers a simplified version of mathematics. Namely, consider strings containing the letters A and B as playing the role of logical statements. In this simplified model, a deduction is a string substitution. The role of axioms is played by a set of string substitutions like {"AB"->"BA","B"->"ABA"}. In this example, starting from "B" one gets all strings of the form "A...ABA..A"— more precisely, any string with one B and an even number of A's, except those with more to the left of the B.

If one takes negation to switch A to B and vice versa, {"A"->"B","B"->"A"}, then this "theory" would be consistent because no string occurs along with its negation. But it would be incomplete since some strings like "ABBA" do not occur in the list of "proved" statements, even in the form of their negation. Note that the deductions only go one way in this model. This model is quite simple, but it already demonstrates phenomena corresponding to inconsistency and completeness.

Many questions remain about, for example, the limits of proof-based methodologies in this model, and to what extent such a model captures the essence of mathematics. These are not just toy questions, but have relevance, for instance to automated theorem proving. Because these questions can take concrete form, so that even statistical tests are possible, it is easier to study the simpler models.

In NKS this idea is applied to various phenomena in science. In the example above one sees the very foundations of mathematics in a simplified setting. It is not hard to see that this approach of computer modeling can be applied to other fields of mathematics as well.

Simple Rules

Let me go back to what is meant by a simple rule. Technically, a simple rule is one with low algorithmic information content. But it is easier to think about simple rules in terms of examples.

One of Stephen Wolfram's early discoveries was the surprisingly interesting behavior of elementary cellular automata: one dimensional with two colors and neighborhoods of size 3. There are precisely 2^{2^3}=256 of these, because there are 2^3 possible neighborhoods plus a rule determining the new color of the middle cell for each of these neighborhoods.

A central agenda of mathematics, and of science in general, is to reduce complicated or confusing behavior by describing it in simple terms, which might take the form of a formula or a collection of axioms and theorems.

One of the innovations of NKS is to reverse this approach: start by considering all simple rules, find out what they do, then apply them as appropriate to real-world observations. Advances in computer technology and software like *Mathematica* (which not only does symbolic computations but also generates computer graphics) have made it feasible to carry out such a project.

An important assertion of NKS is that rules capable of describing natural phenomena, including even very complicated behavior, are often very simple and will turn up in a computer enumeration of the sort described above. The book is full of examples.

Wolfram argues that it is more natural, and less biased, to examine in depth the behavior of rules, prior to guessing which ones might apply to which phenomena. One advantage of surveying possible rules before studying natural world phenomena is that humans tend to embrace preconceived notions of how systems work. Wolfram's principal illustration is the assumption that complicated behavior will require complicated descriptions. It is easy to make such mistakes, even for those who in principle know better.

Polynomials

When a new approach is proposed, one might ask, "What does this tell me about polynomials?" Addition and multiplication have long been the most fundamental objects of mathematical computing. Some might think that nothing new can be learned about what polynomials do, especially by looking at them from a basic point of view, but it turns out that the ideas of NKS apply to all simple rule systems, even polynomials.

In the 20th century, Matiyasevich and others working on Hilbert's tenth problem found a deep connection between computation and Diophantine equations—namely, that questions about the output of a computation correspond to questions about the solutions to a Diophantine equation, or even the values of a polynomial with integer coefficients.

In particular, there are polynomials for which the question of whether they have integer zeros is undecidable. Related to this discovery are phenomena such as the existence of equations that require large solutions. NKS gives the example that the smallest solution to $x^3+y^3-z^3=2$ is x=1214928, y=3480205, z=3528875.

NKS proposes an intuitive notion of computation as a process, whereas the traditional mathematical definition from computability theory, used for instance by Matiyasevich, is more static (though not completely static). From one point of view, thinking of a polynomial as a process clarifies its interpretation as a computation. The Principle of Computational Equivalence has implications as to the computing power of polynomials. In particular, it suggests that not only is the typical Diophantine equation undecidable, it also represents a universal computation.

An example of an undecidable Diophantine expression can be found on p. 786, based on the idea of encoding rule 110's evolution in a set of equations. As for simple examples, a search turns up some with two and three variables for which current theory cannot say if there is a solution, and which appear to be undecidable. It is still unknown what the simplest undecidable equation is. Based on the intuition gained by his other studies, Wolfram suggests that some of these small simple ones are universal.

Undecidability

It seems obvious that it can be highly useful to know that something is impossible. Yet, generally speaking, undecidability has had little impact on most of mathematics. The intuition that proof-based methods can solve any problem is still widespread. Only in a few fields has undecidability had a significant impact. Notably, it is recognized that the undecidable word problem for finitely presented groups (and its related problems) has implications in geometric group theory and low-dimensional topology. For instance, there can be no effective classification of four manifolds.

Faced with mathematical problems that remain intractable under traditional treatments, we must either give up or embrace new approaches. NKS demonstrates the effectiveness of the experimental approach, based on the idea of simple rules. It also presents suggestions for ways to detect irreducibility, which would be immensely useful. For instance, it might be possible to develop practical tests for whether a mathematical problem is tractable using proof-based methodology.

Moreover, the experiments done in NKS suggest that undecidability is the norm. The book mentions the possibility that undecidability has not been a big factor in mathematics because of the prevailing norm of proof-based methodology. A problem's being tractable, or provable, implies that it is decidable. By selecting problems that seem likely to be provable, mathematics has concerned itself mainly with decidable questions, and in the process has missed a wide range of interesting possibilities.

Logic and Axioms

Not surprisingly, fundamental discussions of mathematics often turn to logic and axioms. NKS devotes about 80 pages (out of 1200) to a discussion of them. Most of the axioms discussed in NKS are in the form of equational logic. For instance, the associative law is written a(bc)=(ab)c. There is a finite collection of equations and it is understood that they must all be satisfied for any substitution of variables. The associative axiom serves as an axiom system for semigroups. For the most part, the axioms in the book do not contain constants (which occur, for example, in group presentations).

One traditional mathematical result in NKS is its demonstration of the shortest possible axioms for propositional logic. The proof was computer generated.

The tables of axiom systems in NKS (e.g., p. 804) could be taken as describing what could possibly fall under the heading of mathematics. Most axiom systems found in the NKS enumeration do not fall under any branch of known mathematics, yet many contain interesting finite models. What they do, and what they could be used for, is for the most part unexplored.

It is impossible to be able to predict, using a machine or a mathematical theory, what these systems do, on a case by case basis, even though many interesting general facts are known from the field of universal algebra. Compared to groups, or even to semigroups, little is known about these other "mathematics".

Computation, Intermediate Degrees, and Applicability

A central conjecture of NKS is that natural processes can be viewed as computations. Wolfram proceeds to define computation accordingly. In particular, a computation can be any ongoing process, not necessarily designed as an algorithm. The output is loosely defined as what you can get out of the process. For instance, a computer program can be modified to output its internal variables during the calculation, and this modified program should be considered equivalent.

This kind of definition raises some difficulties in formalizing what a computation is. One challenge is to understand the role of intermediate degrees, which have been formalized and studied for some time in the field of recursion theory. Roughly speaking, a computation degree is represented by the output of a computing device in the form of a subset of the natural numbers. An intermediate degree is represented by a sequence that is too complicated to be predicted on its own, but too simple to be used to predict the output of a universal device (i.e., a recursively enumerable set that is neither recursive nor complete for recursively enumerable sets).

By the Principle of Computational Equivalence, it immediately follows that a machine that represents an intermediate degree must have internal states that can be interpreted to yield a universal machine. Wolfram discusses this, and mentions that this is true for all known examples. My impression is that intermediate degrees may be represented in the natural world by engineered processes, which by design are to some extent predictable, but usually contain irreducible processes.

When mathematics gets applied to the physical world, issues of correctness and appropriateness always arise. It is not clear how one can test the appropriateness of using simple rules except by actually doing so. While there are many examples in NKS, the range of possible examples is as immense as the whole of science. At the moment, it seems that the best thing to do is to try these ideas on as many different systems as possible.

Structure of Mathematics

The idea that the various objects of mathematics can be used to do computations might seem to be redundant, since one might argue that these objects were designed for the purposes of computing in the first place. And it is well known that there are deep connections between different areas of mathematics. It is also well known that some facts can be understood from multiple points of view, for example, when the same theorem can be proved in several significantly different ways.

The Principle of Computational Equivalence suggests that none of this should be surprising. As certain simple machines turn out to be important for certain tasks, so certain simple mathematical systems, until now largely uninvestigated, may turn out to be important for various scientific and mathematical purposes.

NKS proposes that additional effort could fruitfully be given to what it terms empirical metamathematics, addressing such questions as "What is a deep theorem?" If lemmas and theorems are thought of as vertices in a directed graph, where A is connected to B if A is used in the proof of B, then it becomes possible to investigate whether there is a graph-theoretical definition that would decide whether a theorem qualifies as deep or trivial or elegant, etc.

As one such investigation, Wolfram presents an unusual empirical study of the theorems in propositional logic. On p. 817, in an array of all theorems written out in order of increasing complexity, he highlights the interesting theorems, defined as those that have acquired names and are discussed in textbooks. He notes that the first thirteen such highlighted theorems are precisely the ones that cannot be derived from those appearing ahead of them in this ordering. (The fourteenth, which does not appear until the 2814th position on the list, is derivable from preceding theorems—but only by a long proof.)

History

Among the Notes in NKS are discussions of the histories of topics covered in the main text. Those relevant to mathematics fall into two categories. Some are about relatively little-known subjects, such as the search for the shortest axioms for group theory (p. 1153). Others are essays, such as the one on the history of mathematical notation (p. 1182). Most of the major fields of mathematics are discussed at least to some extent.

Mathematical Physics

Several physical phenomena are studied from the NKS point of view. These could be described as mathematical physics because they are about physics and the models are mathematical in nature.

In NKS, Wolfram conjectures that one can derive all of physics from two ideas: all physical phenomena are events, and their causal connections can be described by simple rules. The model in NKS is a causal network, a directed acyclic graph corresponding to an evolution of a simple substitution system. Conditions are given that ensure causal invariance, so that the causal network is independent of the choice of evolution. Essentially, the causal network is a matter-space-time.

Some things from traditional physics follow almost immediately. For instance, the relevant features of special relativity come from causal invariance, which implies that the physics is independent of the evolution. It follows that the speed of light will be constant, and physical laws will be the same, for any observer. It is not so clear how other familiar features of physics, such as quantum mechanical effects, will emerge. NKS suggests several possibilities, but until the ultimate rules are discovered these details will remain unclear.

Geometry from Discreteness

NKS says that underlying all of space and time is a discrete network. What geometry can be derived when all one has are points and connectivity data? How can this network limit to the observed continuum? Similar questions arise in geometric group theory, from the metric point of view. In the NKS model, there is less regularity. In fact, based on the examples in NKS, one expects that the phenomenon of intrinsic randomness will result in a somewhat chaotic appearance at small scales, but which smooths out at larger scales.

In NKS, the volumes of balls, cylinders and cones are used to define certain characteristics like dimension, and even Ricci curvature. This leads to a description of the vacuum Einstein equations. Many questions remain, including even the right definition of limit in this setting.

Educational Ideas

NKS offers many educational possibilities. Math classes could be built upon almost any section of the book. Some sections would require significant student sophistication but many could serve as bases for a class at any level. Also possible are classes on understanding science from the NKS perspective.

Here is a type of class that I have been thinking about: studying the potential of simple rules is possible at any stage of education, and could also directly benefit any student's understanding of science in general. Because computer graphics are easy and useful, projects investigating simple rules are fun. It is relatively easy for students to comprehend a set of simple rules, and even to construct their own conjectures. They will learn that some conjectures are false, as they discover counter examples. They may be able to prove others to be true. Or they may just accumulate evidence suggesting truth or falsity.

In any case, students can learn a great deal by experimenting with simple rules, and not just about mathematical proof, but also about degrees of confidence in conjecture. A parallel activity for advanced math students might be to discover and prove theorems in an uninvestigated axiom system. Side benefits would include practice in rigorous thinking and experience in the use of computers.

One of the basic themes of NKS is that it pays to delve into subjects by considering simple rules, or simplified versions. In NKS, Wolfram extends this idea to questions outside the scope of natural science, such as free will. An interesting project for students would be to develop their favorite subjects from scratch, using the methods found in NKS. Such projects would require each student to think deeply about the subject.

Indeed, this line of thinking suggests that the very idea of education itself could be examined from an NKS point of view.

Conclusion

I hope this brief note conveys my conviction that NKS presents the mathematics community with many specific questions and possibilities and has significant implications for our entire enterprise. But no commentary can convey the full diversity and depth of mathematical content that a reader will find when confronting the book itself. I am confident that any mathematician can find in NKS abundant material worth pursuing. Its extensive index will be highly useful in locating treatments of or suggestions for fields of particular interest.

Todd Rowland
Wolfram Research, Inc.
rowland@wolfram.com

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