A Comprehensive Mathematics Curriculum with *Mathematica*

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MAA Contributed Paper Session: "The Effective Use of Computer Algebra Systems in the Teaching of Mathematics." Joint Mathematics Meetings, Phoenix, January 7, 2004.

Abstract: This paper reports on a comprehensive advanced mathematics curriculum featuring extensive use of CAS software [*Mathematica*] in a new technology-focused California high school. *Mathematica* is an integral component of courses in precalculus, calculus, advanced calculus, modern finite math, differential equations, linear algebra, and discrete math. We describe the specific content of the curriculum; discuss the philosophy and genesis of the program and explore major curricular and instructional issues; look at examples of student work in these courses using *Mathematica*; and report preliminary results and observations about program outcomes.

Introduction

This paper reports on a comprehensive upper secondary level mathematics curriculum at a new California independent school. The extensive use of CAS software – *Mathematica* – supports the integration of advanced mathematics and active projects-based learning. The program encompasses precalculus, AP calculus and AP statistics, finite applied math, multivariate calculus, differential equations, linear algebra, discrete math, and dynamical systems.

Through this program, students are developing greater understanding and insight into mathematical reasoning, and becoming more fluent with mathematical expression. Courses emphasize student projects which require sustained independent as well as collaborative investigation; develop both mathematics understanding and sophisticated computer skills; teach mathematical presentation and exposition; and provide advanced high school students a substantial immersion into many aspects of contemporary mathematics.

In recent years, new approaches to mathematics education have been suggested. These approaches have emphasized learning mathematical reasoning in the context of modeling real-world situations, writing with mathematics, problem solving, and analyzing and interpreting patterns in data, along with a correspondingly reduced emphasis on purely computational mechanics. This program demonstrates that the use of CAS software can facilitate and enhance the effective application of these ideas to advanced high school mathematics and the elementary undergraduate curriculum.

The syntax and structure of *Mathematica* mirrors the structure and syntax of mathematics. Learning *Mathematica* and learning to use it, to write with it, and to develop and study mathematical models of applied problems provides effective training and exercise in learning to think mathematically. Although CAS software is not the only relevant innovation – extensive web resources and networking technologies are also integral to this program – the greatest impact, and the most significant to a view of the evolving nature of mathematics and mathematics education, is with the use of CAS software.

School Background

The Harker School is an independent, nonsectarian, private day school in San Jose, California. enrolls 560 students in grades 9-12. The high school is 5 years old, having graduated two senior classes to date. The high school program – grades 9-12 – was added to the school's existing and long-established elementary school. Extensive expansion of campus facilities was carried out to make the new school possible. From the outset, the school was to emphasize science, mathematics, engineering, and technology, and feature a comprehensive and extensive integration of instructional technology.

Enrollment is selective and competitive. The extremely diverse student body reflects the demographics of families engaged in the technology and academic industries of Silicon Valley and the San Francisco South Bay area. Most students have one or more parents working in the technology industry, and the school enjoys extensive material support of the community and the technology industry. The majority of students plan careers in science, mathematics, technology, engineering, or related fields, although fine and performing arts have begun to be major areas of high student interest, and extensive programming is developing in this area. Technology is integrated throughout the curriculum and in all disciplines.

Virtually all students proceed after graduation to enroll in competitive universities, and the school has quickly established an exceptionally strong record of college admissions. Mean Mathematics SAT1 for 2002-003 was 706; 58% of the senior class of 2003-2004 was named National Merit semifinalist or commended student. In short, the student population is academically strong and highly motivated. With such a strong and highly motivated student body, and extensive technology and academic resources, the school has developed an ambitious academic program. The curriculum stresses advanced placement courses. In 2002-2003, of a student body of 500 students, 279 students wrote a total of 659 AP examinations, including 109 Calculus AP exams. [See Table 1]

%Scoring 4 or 5 in AP exams	AB Calculus	BC Calculus	AP Statistics
2001 (LiveMath)	5 : 73 % 4 : 23 % N = 22	5 : 80 % 4 : 20 % N = 10	not offered
2002 (Mathematica)	5 : 47 % 4 : 32 % N = 51	5 : 85 % 4 : 9 % N = 33	5 : 21 % 4 : 42 % N = 19
2003 (Mathematica)	5 : 53 % 4 : 26 % N = 70	5 : 79 % 4 : 18 % N = 39	5 : 47 % 4 : 34 % N = 32

Table 1 – Some Mathematics AP Results

Two members of Harker's math faculty received an Education Grant from Wolfram Research to develop a math program integrating the use of *Mathematica* in 2001-2002. Subsequently, the school has developed a close relationship with Wolfram Research. [See http://www.wolfram.com/news/harker.html] Members of the mathematics faculty have consulted with Wolfram on educational marketing and have presented *Mathematica* workshops under Wolfram's sponsorship. Currently, the school is in the third year of using *Mathematica*, which is used in most mathematics classes from Precalculus on. In addition, *Mathematica* is used in Advanced Placement Physics and in Advanced Placement Computer science courses.

Infrastructure

All students in grades 9-12 use laptop computers in school and at home. The use of the laptops is integrated into the curriculum in all disciplines in a variety of ways, including classroom presentation and demonstration, extensive use of web-based information resources, collaborative writing, online conferencing and discussion groups, simulation studies and online data analysis in science, presentation authoring and the use of digital video. Mathematics, science, and visual arts programs are increasingly making use of online special-purpose applets.

The campus is covered by both a wireless network [primarily for student use] and a wired network allowing high speed access to the school's network and to the internet by all students and faculty at all times. Students and faculty also have access to the network from home and off-campus. In addition to student laptop use, computer and electronic information resources are provided in numerous computer and science labs throughout the campus and in the school's library/media center. All faculty use computers regularly for both administrative/management as well as instruction and research. All classes make extensive use of both internet and intranet for posting assignments, distributing resources, and publishing student work.

Students buy and use their own laptops. The school is committed to broad cross-platform usability, and students use Windows machines, Macs, and Linux computers. Minimum standards for computer hardware are developed each year. Students purchase required software, both on their own and through the school store, under academic site licenses and special discount programs negotiated by the school. The school makes use of a broad site license negotiated with Wolfram Research, which provides student software at low cost, and ensures that all students and faculty work with the most up-to-date versions of *Mathematica* at all times.

Training has been a major issue in the adoption and implementation of the *Mathematica*-based math curriculum. All members of the mathematics faculty attend either 2-day training courses conducted by Wolfram Research, or 5-day courses during summer institutes [Colorado Rocky Mountain *Mathematica* Academy] prior to their teaching any course using *Mathematica*. Several mathematics faculty as well as computer science faculty have attended advanced *Mathematica* programming courses. Frequent in-service programs provide for further training and faculty development in *Mathematica*. At least once a year, the mathematics department faculty participate in a departmental seminar during which faculty members make presentations about various aspects of mathematics; many of these sessions are devoted to *Mathematica*.

In June, 2004, Harker will launch an annual summer teaching institute on *Mathematica* and projects-based learning in math and science for upper level high school and college.

Curriculum content

Tables 2 and 3 summarize the course offerings in mathematics.

Although advanced placement is stressed in the overall curriculum design of the school, course offerings in mathematics continue well beyond the standard AP courses. Most students take Calculus – either AB or BC – in the 11^{th} grade year, and a measurable proportion is able to do so earlier. Part of the reason for

Course	Calculator	Laptop Course ★	Software	Notes	Texts
Algebra 1	TI83+				R:PrenticeHall Alg1 H:HoughtonMifflin Alg1 Structure & Method
Algebra 2/Trig Honors & Regular	TI83+			Students usually complete Alg 1 and Alg 2 before Geometry	R:PrenticeHall Alg1 H:HoughtonMifflin Alg2 Structure & Method
Geometry Honors & Regular	TI83+	★ Also In-room computer lab	Geometer's Sketchpad		Jurgensen, *Geometry*
Precalculus Honors & Regular	TI83+	★	Mathematica		Demana, Waits, Foley, Kennedy, *Precalculus* 5^{th} ed.
AP Calculus AB	TI83+ (or 89)	★	Mathematica	AB + C is a 4 semester sequence.	Finney, weir, Giordano, *Thomas'Calculus*, 10^{th} ed Coombes, Lipsman, Rosenberg, *Multivariable Calculus and Mathematica*
Calculus C		★	Mathematica		
AP Calculus BC	TI83+ (or 89)	★	Mathematica	BC + Multivariate is a 3 semester sequence, covering the same material as AB+C	
Multivariate Calculus		★	Mathematica		
Differential Equations		★	Mathematica; DFIELD & PPLANE; others	Advanced honors level	Edwards and Penny, *Differential Equations, Computing and Modeling*, 2^{nd} ed.
Advanced Mathematics	TI83+	★	Mathematica	Non-AP course open to any student after Precalculus	Mason, *Mathematics and Modeling with Mathematica*; Armstrong, Davies, *Brief Calculus*
Advanced Geometry		★	Geometer's Sketchpad; NonEuclid; others	Open to any student after Precalculus	Posamentier, Advanced Euclidean Geometry
Intro Statistics	TI83+ (or 89)	★			Mann, Introductory Statistics, 4^{th} ed.
AP Statistics	TI83+ (or 89)	★	Mathematica		*Practice of Statistics*, Yates, Moore, and stearns, 2^{nd} ed.
Linear Algebra		★	Mathematica		Anton and Rorres, *Linear Algebra* 8^{th} ed. [Applications version]
Discrete Mathematics		★	Mathematica; various applets	Open to any student after Precalculus	Carlisle, fisher, Froelich, 2^{nd} ed.
Adv Topics in Math		★	Mathematica; Chaos for Java; others.	By special arrangement only. Current topic: Dynamical Systems	Hillborn, *Chaos and NonLinear Dynamics*; Davis, *Exploring Chaos*

Table 2. Mathematics Course Offerings

Typical Sequence

Grade 7:	Algebra 1
Grade 8:	Algebra 2
Grade 9:	Geometry
Grade 10:	Precalculus
Grade 11:	Calculus [Advanced Placement, AB or BC]
Grade 12:	Calculus C [If AB] or Multivariate Calculus/Differential Calculus [If BC]

A More Accelerated Sequence

Grade 6:	Algebra 1
Grade 7:	Algebra 2
Grade 8:	Geometry
Grade 9:	Precalculus
Grade 10:	Calculus [Advanced Placement, AB or BC]
Grade 11:	Calculus C [If AB] **or** Multivariate Calculus/Differential Calculus [If BC]
Grade 12:	Linear Algebra; Statistics [Advanced Placement]

An Even More Highly Accelerated Sequence

Grade 6:	Algebra 1
Grade 7:	Algebra 2
Summer:	Geometry [Advanced Placement, AB or BC]
Grade 8:	Precalculus
Grade 9:	Calculus [Advanced Placement, AB or BC]
Grade 10:	Calculus C [If AB] **or** Multivariate Calculus/Differential Calculus [If BC]
Grade 11:	Statistics [Advanced Placement] and/or Discrete Mathematics
Grade 12:	Linear Algebra; Advanced Topics in Mathematics

- Calculus AB + Calculus C is a 4-semester sequence covering the complete scope of elementary single and multivariate calculus.
- Calculus BC + Multivariate Calculus covers the same material in 3 semesters. Students typically take Differential Equations in the remaining semester.
- Besides reflecting the more ambitious College Board AP syllabus in BC Calculus, BC classes are conducted both at a substantially more rigorous level and typically include more, and more demanding, Mathematica-based projects.

Table 3. Typical Sequences of Math Courses

this is the ambitious course of study often adopted by students and parents while in middle school. Students are able to complete Algebra 1, Algebra 2, and Geometry while in middle school, and a summer program of courses allows students to accelerate their studies even further.

Typically about 3/4 [for the current year, 78%] of all students take either AB or BC calculus. Of the students that take calculus before their 12th grade year, experience for the last three years has been that more than 90% [current year, 97%] continue to take more advanced courses beyond the advanced placement calculus. In general, almost half of each graduating class [current year 46%] will have completed one or more advanced level course beyond advanced placement calculus by the time they graduate.

Based partly on these enrollment patterns together with the extensive use of *Mathematica* in calculus and the advanced classes, the school is beginning this year to formalize the outcome goal of having all students achieve a prescribed level of proficiency in using *Mathematica* before graduating. Precisely how this proficiency will be defined is currently a top subject for investigation and faculty planning workshops. [1]

The precalculus course follows a standard high school curriculum. Both honors and regular sections are offered, and *Mathematica* is used in both, along with the TI83+ calculator. The faculty is currently engaged in developing content and proficiency standards and goals for *Mathematica* at this introductory level.

Standard curricula exist for AB and BC Calculus and for AP Statistics, although the school's calculus courses tend to be more rigorous than most AP calculus courses. *Mathematica* is used extensively throughout the calculus classes, for problem solving, both in class and on assignments; for presentation and demonstration purposes in lecture; and for extended student projects and investigations. AP results for the last two years do not reveal any apparent drop-off in AP scores as a result of the emphasis on *Mathematica* in the classes. [See Table 2.] However, the data are too limited at this stage – and no controlled study has been attempted – that would justify any claim that using *Mathematica* has had any impact, either way, on AP performance.

For the rest of the program, no national standards or curriculum exists at the high school level. The math faculty has been concerned that the advanced course offerings satisfy rigorous standards; provide students with strong backgrounds for further study – either repeating the corresponding courses in college or actually waiving them; and have a measure of credibility.

The multivariate course was designed in parallel with the Multivariate Calculus at Harvard University for math and physics majors; linear algebra was loosely based on the corresponding course at MIT; and differential equations drew heavily from the corresponding course at Stanford University. Final examinations in these courses, for example, have drawn on examinations in these respective university courses. [The choice of these university courses was based on the rigorous level of the courses and the availability of the course materials.] No other effort has been done to gain "certification" of the course results, but in California, high school courses are routinely submitted to the University of California for certification. Early anecdotal reports from graduates who have begun study in math and science at the university level indicates they are more than satisfied with the preparation these courses gave them. As an increasing number of high schools around the country offer advanced courses at this level, there does need to be developed some national system of formally certifying the course content and the course outcomes.

Genesis and Development of use of *Mathematica* in the Curriculum

The high school program was inaugurated in 1998-1999 with a small 9th grade class. Thereafter, a grade was added each year, as well as increasing the size of each of the classes added. Although the commitment to a technology-centered curriculum was a founding principle, this had little impact on the mathematics curriculum for these first two years.

The laptop program, requiring students to have their own laptop computers for use in school and at home, was launched in 2000-2001, the first year an 11th grade was enrolled. During that year, the math department used the LiveMath CAS program in precalculus and calculus classes. The experience was generally unsatisfactory. Although the program featured a not-insignificant learning curve, it was at the same time judged not robust or powerful enough. In addition, the concept of a CAS program was, at the time, too new for both students and faculty. Few teachers had any background in the use of a CAS, and at the outset only one had any knowledge of LiveMath.

The LiveMath experience, however, addresses some of the significant issues in assessing the role of CAS software in mathematics education.

Mathematica was considered for adoption at the time the decision to go with LiveMath was made. At that time, *Mathematica* was in version 3. The *Mathematica* Front End still feature relatively obscure and difficult syntax as the only way to communicate with the program. Had version 4 been available at that time, *Mathematica* may well have been adopted a year earlier. Subsequent experience has borne out, it is believed, the perception that the availability of a reasonably friendly and accessible user-interface, including one that uses natural and traditional mathematical notation, is a vital ingredient in successfully using the software in an educational setting.

LiveMath was also adopted in part out of the [erroneous] belief that a high school program would be more well-suited to a moderately powerful CAS program with more limited applicability, rather than the most full-featured high-powered CAS. Indeed, LiveMath had not insignificant strengths, including very powerful and flexible function graphing routines in 2 and 3 dimensions. However, the LiveMath experience supported the “All or nothing” argument, and the decision was made to adopt one of the full-power CAS programs on the market for the following year. A major reason for this – The “Complete working environment” argument - is discussed later in this paper.

Mathcad, Matlab, and Maple, were also considered, with Maple and *Mathematica* getting the most extensive evaluation. The decision to adopt *Mathematica* was made after key members of the department were trained in *Mathematica* by Wolfram Research. Two members of the faculty received an Education Program Grant from Wolfram to develop and implement the curriculum using Mathematica, and this became the final factor that pushed the choice to Mathematica.

It should also be noted that the decision to adopt *Mathematica* was made after Version 4.0 of *Mathematica* was released. The existence of the powerful *Mathematica* Front End, which greatly facilitated the user interface and the authoring of original notebooks, was a compelling feature.

Support for *Mathematica* implementation in 2001-2002 was provided by the grant from Wolfram. Subsequently, the school negotiated and developed with Wolfram an unlimited student site license, which was implemented during the 2002-2003 year. Currently, the school is in the second year of the 3-yr license agreement, the third year of using Mathematica, the fourth year of using a CAS in upper level courses.

Why do we Believe *Mathematica* Enhances Teaching and Students’ Learning?

A number of principles and philosophical beliefs have driven our decisions to use a CAS extensively and to adopt Mathematica.

■ Technology is altering the shape/content of the discipline of mathematics [and conversely]. We see evidence for this, for instance, in the following.

- The development of electronic libraries of interactive mathematical knowledge

- The *algorithm* , along with the function, as a new fundamental paradigm of mathematical thinking
- Movement *from* Analysis as a foundation of useful mathematics *to* Discrete Math, Statistics, and Algebra
- Resurgent interest and developments in Linear Algebra, driven largely by applications in digital sciences
- Interest in dynamical systems, linear/nonlinear/chaotic
- The growing emphasis on the construction, analysis, and use of mathematical models
- A convergence of numerical and symbolic problem-solving, facilitated by enhanced technical computing power
- New applications, and renewed interest in applied mathematics across the curriculum

■ CAS programs like *Mathematica* - in structure and syntax - mirror the structure and syntax of Mathematics. We can expect that learning a CAS will *automatically* reinforce ways of thinking mathematically. Examples of where this happens include:

- The behavior and properties of numbers
- The structure of expressions
- Lists/arrays/matrices/tables
- Functions: defining/analyzing/evaluating

■ Technology facilitates working with cumbersome and tedious computation and representation to allow more complex problems, more real problems, and allows students to "see the forest for the trees." Students who have used *Mathematica* for at least a year identified this as one of the two most significant reasons for using the CAS and why they believe it has helped their understanding of mathematics.

■ Projects: Students *learn* mathematics by *doing* mathematics. In a fundamental sense, the adoption of a "computer algebra system" was not our fundamental educational objective [important to keep in mind in this day and age when many schools are driven to adopt technology for the sake of technology!]. Our fundamental aim was to implement a curriculum that emphasized *project learning*.

> Most applications of mathematics in the real world are multifaceted and complex. Often, just understanding the problem is the most challenging part of solving it. Rarely do we find in real world problems the scenarios usually encountered in mathematics texts, simple relationships, typically accompanied by a small number of inputs and a single "correct" output or answer. More often, solving problems using mathematics requires sustained work, bringing to bear multiple

resources, using the collaborative efforts of a "problem engineering team." How the problem is stated and described becomes important, as does how the solution is presented. And rarely is the "solution" the end. Solutions must be interpreted, conclusions and inferences drawn, and solutions recycled into a refined understanding of the problem.

Fundamentally, *Mathematica* was adopted because of the belief that it would facilitate a projects-oriented curriculum. In the school's math program, increasingly as the level moves upward, students use *Mathematica* to carry out and write on substantial projects. Projects require sustained investigation over an extended period of time, often need additional research, and often involve non-routine problems. Projects are often done collaboratively. Most importantly, however, projects require substantial *project reports*, most often *Mathematica* notebooks, which present [in good mathematical style] an extended report of the student's work, solutions, and reasoning. Such projects also help students develop better problem-solving skills. And writing the reports – with the aid of a computing tool expressly designed for such a purpose – will help students develop a better command of mathematical style. This will be touched on again later in connection with the "CAS vs. complete environment" argument.

■ Visualization is a critical and often difficult dimension or learning and doing mathematics. Combining the power of a CAS like *Mathematica* and the resolution and graphics engine now accessible in even relatively inexpensive laptop computers makes possible a degree of visualization heretofore impossible and helps to develop graphical reasoning and graphical intuition. More importantly, it takes the graphics out of the textbook and puts them under the control of the student, who can manipulate, investigate, explore. Graphics are always helpful in learning, but we find it makes a difference whether student interaction with graphic visualization is *active* or *passive*. Students at the school who have used *Mathematica* at least one year identified this – visualization – as the other of the two significant benefits they saw in the use of Mathematica.

■ The decision to adopt *Mathematica* was motivated by the desire to prepare students for demanding university level work in mathematics and associated fields. Learning and using *Mathematica* helps to build problem solving skills in the context of problems more like those the student is likely to actually encounter and needing to use mathematics. The vast majority of college courses that students will take in math and in mathematical sciences will require the use of computers, either with *Mathematica* or a comparable program. Many such courses at the same time either discourage or forbid the use of calculators. Moreover, much as learning any computer programming language builds computer skills, learning *Mathematica* should make it easier for students to learn another CAS program later if necessary.

How is *Mathematica* Used?

By Students

- Projects: 100% of students enrolled in math classes in calculus and beyond use *Mathematica* on projects, both for developing solutions and preparing reports

- Homework, routine problem solving: About 25% of students say they use *Mathematica* from time to time, on their own, without it being assigned, to help with homework

- Homework, concept exploration: About 10% of students say they use *Mathematica* outside of routine problem solving for exploring concepts and ideas when it is not specifically assigned

- Programming, "play," on their own: An estimated 10-15% of students use *Mathematica* for programming, for other classes, for their own independent "play." This will certainly reflect the high proportion of students enrolled who have are interested in computer technology.

By Faculty

- Most [not all!] teachers in the *Mathematica*-courses – precalculus and on, say they use *Mathematica* at least once a week in lecture, As a presentation aid, for demonstration or for exploration. Some of this use is spontaneous, facilitated by the ubiquity of computers, networking, and video projection equipment in every classroom

- About 50% of math teachers using *Mathematica* use it from time to time to prepare tests, handouts, overheads, etc.

- A small minority of teachers use *Mathematica* as a tool in researching and in investigating mathematical information resources, such as those at Mathworld and elsewhere.

An Emerging Use

- In some classes, students are beginning to be comfortable presenting work to the class using Mathematica, rather than simply submitting printed or electronic versions of their projects.

Other Student Observations

Asked if they "like" Mathematica, student responses cluster around 3-4 on a 7 point scale [7=high]. Asked if they think *Mathematica* has enhanced their understanding of Mathematics, student responses cluster around 5. It appears – we can say this with some confidence – that it takes a full year for students to become comfortable with *Mathematica* and to *begin* to develop some fluency with it. Students' first reaction to *Mathematica* is almost always, "Huh?" Their second reaction is, "What's the point?" After a year, students begin to be more confident about using *Mathematica* to solve problems and write reports. The level of rigor and the sophistication of the notebooks they write shows a marked increase after a year.

Although not tested, we might conjecture based on the evidence so far that there is a correlation between using *Mathematica* and computer programming in general. Students who do not like to program computers, or who have never learned a general purpose programming language, seem to have the least enthusiastic response to Mathematica.

There seems to be no correlation – although this has not been tested either – between a student's use of *Mathematica* and the student's mathematical ability. Some extremely gifted mathematics students have used it to extend the depth and reach of their work significantly; other students with very high ability do not like *Mathematica* and use it as little as possible.

A significant dichotomy in student reactions has to do with preferred mode of learning Mathematica. Almost exactly 50% of students say they would prefer to have *Mathematica* taught, directly, at the outset, in classes dedicated to covering *Mathematica* fundamentals. On the other hand, 50% of students say they would prefer to "pick up" *Mathematica* as they go along, as relevant applications are encountered, acquiring knowledge of *Mathematica* on a "just-in-time" basis.

What does our Use of *Mathematica* Say about the Evolving Nature of Mathematics?

Programs like *Mathematica* have significantly altered the landscape of mathematics and changed the profile of the mathematical skills that need to be acquired by students. For many students, an emphasis on *using* mathematics is more practical and beneficial than the more abstract "pure" aspects of mathematics. Although it is arguable whether one can intelligently *use* mathematics without *understanding* mathematics, or whether the use of a CAS can undermine such understanding, it does appear that the use of programs like *Mathematica* have expanded the range of applicability of mathematics to a wider range of students.

Moreover, as the volume and complexity of the knowledge our society produces and uses continues to grow exponentially, it is becoming impossible for anyone to "know" everything that is important, impossible to master a discipline in the ways we traditionally defined it. The knowledge that we will generate, on top of the vast knowledge developed to date in fields like mathematics, will increasingly "live" in digital libraries and intelligent information databases. We will need to use the associated electronic tools to access and apply our knowledge. *Mathematica* is one such tool. *Mathematica* "knows" an amazing amount of mathematics. Traditionally, we think of a "tool" as something that enhances, extends, or multiplies our natural and unaided ability. *Mathematica* is a tool that enhances our ability to access and use mathematics. We think of helping students master the foundations of applied problem solving by using digital tools like *Mathematica* as a way of extending their native capacity to understand and resolve complex problems. *What is most significant, though, is that the new computational and problem solving tool is also the language or framework with which mathematical knowledge is collected, organized, stored, and retrieved.*

One of the most powerful outcomes is the creation of a new paradigm, namely the *interactive document*. *Mathematica* creates the ability to prepare comprehensive, organized and structured mathematical documents that are – by design - interactive. A user can test various initial conditions, modify parameters and assumptions, evaluate different data, and explore the conclusions and inferences of a dynamic mathematical model. This is related to the notion of a "dynamic mathematical model," something that seems to be central to doing mathematics with *Mathematica.*

Does this mean that the traditional mathematical paper, the ordinary channels of publishing and disseminating new mathematical knowledge will soon be archaic and pointless? One doubts it. But certainly changes are coming, and the students who have learned mathematics from an early stage with *Mathematica* or other CAS programs will understand and relate to new forms of communicating mathematics.

A Comment on the "CAS vs. Complete Computing Environment" Argument

Wolfram Research repeatedly stresses in its marketing and program documentation that "*Mathematica* is not a CAS." Rather, *Mathematica* is a "complete programming environment."

This notion makes a lot of sense in the context of our experience using *Mathematica* in teaching and learning mathematics. As noted above, the problem solving and computational potential of *Mathematica* was not the primary reason the school adopted Mathematica, but rather the fact that *Mathematica* would facilitate a projects-focused curriculum. In this context, it is important that students learn to write mathematics, be able to present it in writing and orally, develop some sense of mathematical style and the correct and fluent use of mathematical notation. The investigation of a complex and multifaceted applied math problem leads to a project report that integrates not only symbolic and numeric computation, but also organized and structured expository text and clear, helpful, possibly "live" graphics. Moreover, the report – the "interactive document" described above – needs to be saved, cataloged and indexed with other work, emailed or otherwise submitted, published or posted. All of these things happen within the *environment* created by using the *Mathematica* program.

Thus, in our experience, we would agree with Wolfram. The *most* significant impact of *Mathematica* on teaching and learning mathematics in our program has come not because *Mathematica* is a CAS, but because it provides an *integrated environment for doing and thinking about mathematics.*

Examples of Projects and Problems

Most *Mathematica* notebooks that reveal the nature of work being done by students are too large for inclusion in this paper. Following are a few illustrative examples of problems that have been the basis for Mathematica-based projects [although a listing of this nature tends to overlook the many less ambitious uses of *Mathematica* in these classes.

Precalculus

- Transformations of functions can be studied using *Mathematica* to develop illustrative graphs. Students are asked to develop notebook-based presentations on transformations of functions and graphs.
- Use *Mathematica* to explore trends in population data... Fit the data to a model [In your report, discuss your attempts to explore and evaluate alternative models], and produce clear and well-labeled graphs of your models with the data. Write interpretations of the graphs and any inferences or observations you can make about the data from your analysis.
- [On Valentine's Day] Use *Mathematica* and polar graphing methods. Find functions whose graphs you can combine to produce interesting and appealing designs for valentine's day cards.
- The Rental Car Problem: Determine gasoline consumption on a drive with rental car from San Diego to San Francisco for various configurations of stops and other parameters. Determine an optimum fueling strategy for the rental car. [Has extensions to calculus, and then to multivariate calculus.]

Calculus

- Analyze the power curve of a baseball swing. Determine the optimum moment of impact with the ball. Be sure to discuss the assumptions of your model as well as conclusions.
- Develop an animation which shows the changing tangent line to the graph of any given differentiable function as the point of tangency moves along the curve.

- Develop a "virtual" [3D] graphic realization on the computer of the vase on the teacher's desk. Use methods of calculus to analyze such quantities as surface area [inner and outer], volume contained, volume of clay, etc. Find a function that will generate the vase by rotation and do it on the computer.

Multivariate Calculus

- A shark swims in the direction of the greatest rate of increasing concentration of blood in the water. Given this distribution function for the blood concentration in the water, apply methods of multivariate calculus to study the path taken by the shark. Produce graphs in various formats and find suitable functions/equations for the path[s]. For extra credit, assume a time-varying 3p0-dimensional blood field.
- Develop a model for the power output of a hydropower generating station with three turbines of different characteristics and varying rates of water flow. Determine the optimum distribution of water flow between the turbines under different configurations and conditions.
- Develop a model using *Mathematica* for the [classical] gravitational fields generated by n bodes in the discrete case, and by distributed mass in the continuous case. Produce graphs that can be used to illustrate the n-body gravitational field. Explore the meaning of the gravitational potential function.

Differential Equations

- *Mathematica* [along with DFIELD and PPLANE] is used extensively and constantly in Differential Equations. The text used [Edwards and Penny] is ideally suited to a technology-based advanced math course. An example of an applied project is the following, drawn from an Edwards and Penny problem.
- Earthquake analysis: With the data supplied about restoring forces for each of the floors, perform an analysis of a seven story building. Determine the resonant frequencies of the floors and the building. Determine the earthquake frequency ranges most likely to produce severe damage. Provide graphs to illustrate. Then determine the same data [see the Plant and Engineering Department] for Dobbins Hall and perform the same analysis.
- Students gain a deeper understanding of such things as transient and steady state solutions, and the relationships between time graphs and phase graphs by using *Mathematica* top solve many problems. They are able to do many more problems, and patterns become more apparent.

Advanced Math, Discrete Math, Statistics

- Markov Chain Problems are especially well suited to Mathematica-based computer projects. Complicated and realistic problems with many variables can be done using Mathematica.

Examples of Mathematica Notebooks by Students

- Advanced Math/Precalculus in-class test with student solutions: Linear Regression problem

- Calculus: The Vase Problem [see above]

- Calculus: Simple tangent line animation

- Differential Equations: Damping and resonance studies
- Multivariate Calculus: Student project including critical point analysis of exotic surfaces, and HydroPower Plant problem – Optimization by Lagrange Multipliers with multiple constraints
- Multivariate Calculus: Student project including shark problem, analysis of surfaces, proof of least squares algorithm, oceanographic surface analysis

These *Mathematica* notebooks can be downloaded from http://www.harker.org/mathematica/. The text of this paper, and links to additional resources, will also be available.

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[1] We will post the "Scope and Sequence" for Mathematica proficiency on the website – http://www.harker.org/mathematica/ - as soon as a draft becomes available. At present, this is planned for an upcoming February faculty workshop.